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SPLIT NEIGHBORHOOD NUMBER IN FUZZY GRAPHS

Arun Kumar A

Department of Mathematics, Sri Mahadeshwara Government First Grade College, Kollegala-571440, India

Email: akumara78@yahoo.co.in

ABSTRACT

In the present manuscript, we present and analyze the notion of a split neighborhood set in the context of fuzzy graphs. Additionally, we delve into the connections involving the split neighborhood number $\eta_s(G)$ with the other known parameters in G. Furthermore, we derive numerous inequalities for the parameter $\eta_s(G)$. Additionally, we provide a result akin to the Nordhaus-Gaddum theorem pertaining to this parameter.

Key words: Fuzzy graph, neighborhood number, split neighborhood number, domination number.

1. INTRODUCTION

In 1736, Euler introduced the foundational ideas of graph theory. Notably, Euler's resolution of the Königsberg bridge problem stands as the inaugural theorem in graph theory's historical context. This field has since evolved into a distinct branch of combinatorics. Graph theory provides a versatile tool for addressing combinatorial challenges in various domains such as geometry, number theory, algebra, topology, optimization, operations research, and computer science. Sampathkumar (1985, 1989) pioneered the definitions of novel concepts: neighborhood number η(G), global domination number $\gamma_g(G)$, and global neighborhood number $\eta_g(G)$, Extending this work, they established multiple bounds for η(G), $\eta_g(G)$, and $\gamma_g(G)$, and explored these parameters across several common classes of crisp graphs. The formal mathematical characterization of domination was attributed to Ore (1962). Cockayne and Hedetnieme (1977) contributed a comprehensive survey on this subject. Zadeh (1965) introduced the groundbreaking concept of fuzzy set theory. Expanding on this notion, Rosenfeld (1975) introduced fuzzy graphs along with various fuzzy analogs of graph theoretic principles like paths, cycles, and connectedness. Zadeh (1987) introduced fuzzy relations. Bhattacharya (1987) and Bhutani (1989) delved into fuzzy automorphism graphs. Mordeson (1993) extended this concept to fuzzy line graphs and developed fundamental properties. Somasundaram (1998) investigated domination within fuzzy graphs. Mahioub and Soner (2007) explored neighborhood numbers, covering numbers, and global domination numbers in fuzzy graphs. In the scope of this paper, we introduce and scrutinize the split neighborhood number in the context of fuzzy graphs.

2. Preliminaries

We briefly review some definitions in crisp graphs and fuzzy graphs. Notation and terminologies of our work are found in [4,9].

A crisp graph G is a finite non-empty set of objects known as vertices, coupled with a set of unordered pairs of distinct vertices of G, termed edges. The sets of vertices and edges of G are denoted as $V(G)$ and $E(G)$, respectively. In a crisp graph G, a subset S of vertices is referred to as a neighborhood set if G is equal to the union of all subgraphs $\langle N[v]\rangle$, where $\langle N[v]\rangle$ represents the subgraph of G induced by v and all vertices adjacent to v. The neighborhood number $\eta(G)$ of G is the smallest cardinality of a neighborhood set in G. A neighborhood set S of a crisp graph $G = (V, E)$ is considered a split neighborhood set (sn-set) if the subgraph ⟨V-S⟩ induced by V-S is disconnected. The split neighborhood number $\eta_s(G)$ of G is the minimum cardinality of an sn-set in G.

A fuzzy graph G is a mathematical structure represented as $G = (\mu, \rho)$, where $\mu: V \to [0,1]$ and $\rho: E \to [0,1]$ are two functions satisfying the condition $\rho(\{x, y\}) \leq \mu(x) \wedge \mu(y)$ for all $x, y \epsilon V$. We use the notation $\rho(xy)$ to represent $\rho(\{x, y\})$. The order of a fuzzy graph G, denoted as p, is defined as be $p = \sum_{x \in V} \mu(x)$ and and its size, denoted as q, is defined as $q = \sum_{xy \in E} \rho(xy)$.

A fuzzy graph $H = (\lambda, \tau)$ be considered a partial fuzzy subgraph of $G = (\mu, \rho)$ if $\lambda \subseteq \mu$ and $\tau \subseteq \rho$. Similarly, the fuzzy graph $H = (D, \lambda, \tau)$ is known a fuzzy subgraph of $G = (V, \mu, \rho)$ induced through D if $D \subseteq V$, $\lambda(x) = \mu(x)$ for all $x \epsilon D$, and $\tau(x, y) = \rho(x, y)$ for all $(x, y) \epsilon D \times D$.

3. SPLIT NEIGHBORHOOD SET

Definition 3.1: In the context of a fuzzy graph $G = (V, \mu, \rho)$, a neighborhood set S is termed a split neighborhood set (sn-set) of G if it results in a fuzzy subgraph represented as $H = (< V - S\rangle, \mu', \rho')$ induced by V-S that is disconnected. The smallest fuzzy cardinality among all split neighborhood sets in G is known as the split neighborhood number of G and is denoted as $\eta_s(G)$ or simply η_s.

Hereafter, we denote a fuzzy graph by FG.

If $\eta_s(G) = |S| = \sum \mu(v), \forall v \in S$, the S is called minimum split neighborhood set of G and say $\eta_s -$ set.

Example 3.1: Consider a FG , $G = (V, \mu, \rho)$ given in Figure 3.1, where $V = \{v_1, v_2, v_3, v_4\}, \mu(v_1) = 0.1$, $\mu(v_2) = 0.25$, $\mu(v_3) = 0.2$, $\mu(v_4) = 0.15$, and $\rho(u, v) = \min\{\mu(u), \mu(v)\}, \forall (u, v) \in \rho^*$.

A subsets $A_1 = \{v_1, v_3\}$ and $A_2 = \{v_2, v_4\}$ are split neighborhood sets in $G, |A_1| = \mu(v_1) + \mu(v_3) = 0.1 + 0.15 = 0.25$ and $|A_2| = \mu(v_2) + \mu(v_4) = 0.25 + 0.2 = 0.45$. Hence, $\eta_s(G) = \min\{0.25, 0.45\} = 0.25$.

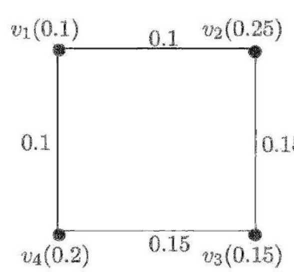

Fig. 3.1

A FG, $G = (\mu, \rho)$ is called star FG if, $G^* = (\mu^*, \rho^*)$ is a star and is denoted by $K_{\mu(u),\mu(v_i)}$.

In the following proposition we give without proof the exact values of η_s for some standard FGs.

Example 3.2:

(i) Since for any vertex sub set S of V in a complete FG, $G = K_\mu$, a fuzzy subgraph induced by $V - S$ is connected, then the only split neighborhood set in G is $V - \{v\}$ for all $v \in V$, then $\eta_s(G) = \gamma_s(G) = p - t,\ t = \max\{\mu(v), \forall v \in V(G)\}$. i.e $t = \mu(v): d_N(v) = \delta_N(G)$.

(ii) If $G = K_{\mu_1,\mu_2}$, a bipartite fuzzy graph, then $\eta_s(G) = \min\{|\mu_1|, |\mu_2|\}$.

(iii) $\eta_s(G) \le p$ if and only if $\rho(uv) = 0$ for all $u, v \in V(G)$. In particular $\eta_s(K_\mu) = p$

Theorem 3.1: For any FG= $(V, \mu, \rho), \eta_s(G) \le p - t, t = \max\{\mu(v), \forall v \in V(G)\}$; further, equality holds if $G = K_\mu$.

Proof: Let $G = (V, \mu, \rho)$ be a FG, it is clear that V is a neighborhood set of G, and $V - \{v\}$ is a split neighborhood set of G. Hence $\eta_s(G) \le |V - \{v\}| = p - t$. Equality holds by example (3.1).

Corollary 3.1.1: Let $G = (V, \mu, \rho)$ be any FG. If G is a complete , then $\eta_s(G) = \delta_N(G)$.

Corollary 3.1.2: Let $G = (V, \mu, \rho)$ be any FG. If G is a bipartite FG or complete bipartite, then $\eta_s(G) = \delta_N(G)$.

Theorem 3.2: In any FG, $G = (V, \mu, \rho)$, every split neighborhood set of G is a neighborhood set.

Proof: Let $G = (\mu, \rho)$ be any fuzzy graph, and let $S \subseteq V$ is a split neighborhood set of G, a fuzzy subgraph $H = (< V - S\rangle, \mu', \rho')$ induced by $V - S$ is disconnected, then $\rho'(w, u) = 0$, for some $(w, u) \in \rho'^*$, and $\rho(u, v) > 0, \forall u \in S, v \in V - S$, then $G = \cup_{u \in S} (< [u] >, \mu', \rho')$. Hence S is a neighborhood set of G.

A consequence of the above theorem we state without proof the following theorem.

Theorem 3.3: In any FG, $G = (V, \mu, \rho), \eta(G) \le \eta_s(G)$; further, equality holds if $G = K_\mu$.

Corollary 3.3.1: For any bipartite FG, $G = (V, \mu, \rho)$ without isolated vertices, $\eta_s(G) = \eta(G)$.

Theorem 3.4: In any FG, $G = (V, \mu, \rho), \gamma_s(G) \le \eta_s(G)$;

further, equality holds if $G = K_\mu$ or $G = K_{\mu_1,\mu_2}$.

Proof: Consider a fuzzy graph G=(μ,ρ) defined on a vertex set V, and let S be a split neighborhood set of G. According to Theorem 3.2, S qualifies as a neighborhood set of G. As every neighborhood set inherently acts as a dominating set, S also functions as a dominating set of G. Since the fuzzy subgraph $H = (< V - S\rangle, \mu', \rho')$ induced by $V - S$ is disconnected, S is confirmed to be a split dominating set of G. Therefore, $\eta_s(G) \ge \gamma_s(G)$. In cases where G equals K_μ or G equals $G = K_\mu$ or $G = K_{\mu_1,\mu_2}$,, as exemplified in Example 3.1, the equality holds.

Theorem 3.5: In a fuzzy graph $G = (V, \mu, \rho)$ with no isolated vertices, it holds that $\eta_s(G)$ is less than or equal to $p - \beta(G)$.

Proof: Consider a fuzzy graph $G = (\mu, \rho)$ defined on the vertex set V, and let D be an independent set in G with a cardinality of $|D|$ equal to $\beta(G)$. In this scenario, either $\rho(u, v) < \mu(u) \wedge \mu(v)$ holds for all $(u, v) \epsilon D$, or there exists some $(u, v) \epsilon D$ for which $\rho(u, v) = 0$, and additionally, $\rho(u, v) > 0$ for all $u \epsilon D$ and $v \epsilon V - D$. Consequently, the fuzzy subgraph $H = (\langle D\rangle, \mu', \rho')$ induced by D becomes disconnected. As a result, $V - D$ can be identified as a split neighborhood set of G. Therefore, it follows that $\eta_s(G) \leq |V - D| = p - \beta(G)$.

Corollary 3.5.1: In the case of any fuzzy graph $G = (V, \mu, \rho)$ assuming no isolated vertices are present, in the scenario where G is either a bipartite fuzzy graph or a complete bipartite fuzzy graph, we have $\eta_s(G) \leq \alpha(G)$.

Proof: Since $\alpha(G) + \beta(G) = p$, then by the above theorem, $\eta_s(G) \leq \alpha(G)$.

Corollary 3.5.2: Consider any fuzzy graph $G = (V, \mu, \rho)$ without isolated vertices. If G happens to be either a bipartite fuzzy graph or a complete bipartite fuzzy graph, then $\eta_s(G)$ is equal to $p - \beta(G)$.

Corollary 3.5.3: In the case of any fuzzy graph $G = (V, \mu, \rho)$ without isolated vertices. If G is a bipartite fuzzy graph or complete bipartite fuzzy graph, then $\eta_s(G)$ is equal to $\alpha(G)$.

Definition 3.2: Consider a fuzzy graph G=(μ,ρ) defined on a vertex set V, with $u \epsilon V$. Then

(i) If $\Delta_N(G) = d_N(u)$, then $|N(u)| = \Delta_N(G)$.

(ii) If $\Delta_E(G) = d_E(u)$, then $|N(u)| = \Delta_E(G)$.

Theorem 3.6: For any fuzzy graph , $G = (V, \mu, \rho), \eta_s(G) \leq \Delta_E(G)$;

further equality holds if $G = K_\mu$ or $G = K_{\mu_1,\mu_2}$.

Proof: Consider a fuzzy graph $G = (\mu, \rho)$ defined on the vertex set V, and let D be a split neighborhood set of G. Consequently, a fuzzy subgraph $H = (\langle V - D\rangle, \mu', \rho')$ induced by V-D becomes disconnected. Now, assume there exists a vertex $v \epsilon V$ such that $d_E(v) = \Delta_E(G)$), and let $u \notin N(v)$. It follows that u must belong to the neighborhood set of some vertices in N(v). This, in turn, leads to a disconnected fuzzy subgraph $H = (< V - N(v)\rangle, \mu', \rho')$ induced by $V - N(v)$. Consequently, $N(v)$ qualifies as a split neighborhood set of G. Therefore, $\eta_s(G) \leq |N(v)| = \Delta_E(G)$.

Corollary 3.6.1: For any FG, $G = (V, \mu, \rho), \eta_s(G) \leq \Delta_N(G)$.

For a split neighborhood number $\eta_s(G)$ The following theorem gives a result of the Nordhaus Gadam type:

Theorem 3.7: Consider any fuzzy graph $G = (V, \mu, \rho)$ where both G and its complement, $\bar{G}$, are connected. In this case, we have $\eta_s(G) + \eta_s(\bar{G}) \leq 2p$, and equality is achieved if and only if ρ(u,v) = 0 for all $u, v \in V$

Proof: The inequality is established by definition. Now, $\eta_s(G) + \eta_s(\bar{G}) = 2p$ if and only if the singular split neighborhood set in both G and in $\bar{G}$ is V, which occurs if and only if ρ(u,v) = 0 for all $u, v \in V$.

Theorem 3.8: In the case of any fuzzy graph $G = (\mu, \rho), \eta_s(G) \geq \gamma(G)$.

Proof: Consider a fuzzy graph $G = (\mu, \rho)$ defined on the vertex set V, and let S be a split neighborhood set of G. As per Theorem 3.4, S is identified as a split dominating set. Furthermore, since every split dominating set inherently serves as a dominating set, S can be classified as a dominating set. Consequently, we can conclude that $\eta_s(G) \geq \gamma(G)$.

Theorem 3.9: In the case of any fuzzy graph $G = (\mu, \rho)$ such that $G \neq K_\mu$, $\eta_s(G) \leq p - \Delta_N(G)$.

Proof: Consider any fuzzy graph $G = (\mu, \rho)$ defined on the vertex set V, and let D be a split neighborhood set of G. As a result, a fuzzy subgraph $H = (\langle V - D\rangle, \mu', \rho')$induced by V-D becomes disconnected. Now, suppose there exists a vertex $u \in V - D$ such that that $d_N(u) = \Delta_N(G)$. According to Corollary 3.6.1, N(u) contains D, implying that $N(u) \cup D$ is a subset of V. Consequently, the cardinality of $N(u) \cup D$ is less than or equal to the cardinality of V. Hence, it follows that $\eta_s(G) \leq p - \Delta_N(G)$.

Corollary 3.9.1: In the case of any fuzzy graph $G = (V, \mu, \rho)$ such that $\neq K_\mu, \eta_s(G) \leq p - \Delta_E(G)$.

Proof: Since $\Delta_E(G) \leq \Delta_N(G)$, then $p - \Delta_N(G) \leq p - \Delta_E(G)$. Hence, $\eta_s(G) \leq p - \Delta_E(G)$.

Corollary 3.9.2: Consider any fuzzy graph $G = (V, \mu, \rho)$. If G is either a bipartite fuzzy graph or a complete bipartite fuzzy graph, then it holds true that $\eta_s(G) = p - \Delta_N(G)$.

Theorem 3.10: In the case of any fuzzy graph $G = (\mu, \rho), \eta_s(G) = \alpha(G)$, if and only, if there exists a split neighborhood set S with $|S| = \eta_s(G)$, such that $V - S$ is an independent.

Proof: If V-S forms an independent set, then S serves as a vertex cover. Therefore, $(G) \leq |S| = \sum_{v \in S} \mu(v) = \eta_s(G)$. Furthermore, according to Corollary 3.5.1, $\eta_s(G) \leq \alpha(G)$ Consequently, we have $\eta_s(G) = \alpha(G)$.

Conversely, if $\eta_s(G) = \alpha(G)$, and S is a vertex cover with $|S| = \eta_s(G)$, then V-S must be independent. Additionally, since every vertex cover is inherently a neighborhood set, and given that $\eta_s(G) = \alpha(G)$,, it logically follows that S is a split neighborhood set with $|S| = \eta_s(G)$.

Theorem 3.11: A set D of vertices in a fuzzy graph $G = (\mu, \rho)$ is considered a split neighborhood set if and only if there exist two vertices u and v in V-D such that in every u-v path, there is at least one vertex from the set D.

Proof: Assume that D is a split neighborhood set of G. In this case, the fuzzy subgraph $H = (< V - D\rangle, \mu', \rho')$ induced by V-D is found to be disconnected. Consequently, we can identify the existence of u and v in V-D such that ρ(u,v)<0, implying that every u-v path includes at least one vertex from the set D.

Now, consider u and v in V-D, ensuring that in every u-v path, there is at least one vertex from the set D. In such a scenario, it follows that ρ(u,v)<0, and the fuzzy subgraph induced by V-D is once again disconnected. Consequently, we can conclude that D is indeed a split neighborhood set of G.

Proposition 3.1: In the case of any fuzzy graph $G = (\mu, \rho)$ is a path such that $(v_i, v_{i+1}) = \mu(v_i) \wedge \mu(v_{i+1}), \forall i = 1,2, \dots, n$, then $\eta_s(G) = \eta(G) = \alpha(G)$.

Proof: Considering that any path is inherently a bipartite fuzzy graph with$(v_i, v_{i+1}) = \mu(v_i) \wedge \mu(v_{i+1})$, for all i=1,2,…,n, it follows that G does not contain isolated vertices. Consequently, with the aid of Corollary 3.3.1 and Corollary 3.5.3, we can establish that $\eta_s(G) = \eta(G) = \alpha(G)$.

Definition 3.3: We refer to a split neighborhood set D of a fuzzy graph $G = (\mu, \rho)$ as a minimal split neighborhood set of G if, for every $v \epsilon D$, the set D-{v} does not qualify as a split neighborhood set of G

Below, we present the definition of $\eta_s(G)$, incorporating the concept of minimal split dominating set, which is equivalent to Definition 3.1

Definition 3.4: The smallest fuzzy cardinality among all split neighborhood sets is denoted as the split neighborhood number of G, represented as $\eta_s(G)$.

Example 3.2: Let G be a fuzzy graph defined in the Figure 3.2. Where $\mu(u) = 0.2, \mu(v) = 0.3, \mu(x) = 0.4, \mu(y) = 0.1, \mu(z) = 0.25$.

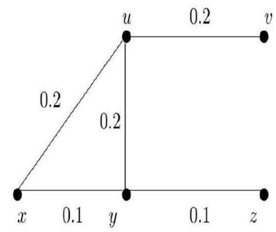

Fig. 3.2

We say that $D_1 = \{u, y\}, D_2 = \{y, v\}, D_3 = \{u, z\}, D_4 = \{x, y, u\}$, $D_5 = \{x, y, z\}, D_6 = \{u, y, z\}$ and $D_7 = \{u, y, v\}$ are minimal split neighborhood set of G. Hence,

$$\eta_s(G) = \min\{|D_1|, |D_2|, |D_3|, |D_4|, |D_5|, |D_6|, |D_7|\}$$
$$= \min\{0.3, 0.4, 0.45, 0.7, 0.75, 0.55, 0.6\} = 0.3.$$

The following theorem gives a characterizations of minimal split neighborhood set of fuzzy graphs .

Theorem 3.12: A split neighborhood set D in a fuzzy graph G=(μ,ρ) is considered minimal if and only if, for every vertex v∈V, one of the following conditions is satisfied:

(1) There exists a vertex $u \in V - D$, such that $N(u) \cap D = \{v\}$;

(2) v is an isolated vertex in $\langle D >$ (i.e; $\rho(u, v) < \mu(u) \wedge \mu(v), \forall u \in D$);

(3) A fuzzy subgraph induced by $\langle V - D \rangle$ is connected.

Proof: Assume that D is a minimal split neighborhood set, and there exists a vertex v∈V that does not satisfy any of the aforementioned conditions. Based on conditions (1) and (2), it follows that $D' = D - \{v\}$ qualifies as a neighborhood set of G. Furthermore, as per condition (3), the fuzzy subgraph induced by $\langle V - D' \rangle$ is disconnected. Consequently, we can conclude that D' also serves as a split neighborhood set of G, which leads to a contradiction.

Definition 3.5: In the context of a connected fuzzy graph $G = (\mu, \rho)$, a neighborhood set D is termed a connected neighborhood set of G if the fuzzy subgraph $H = (< D >, \mu, \rho)$ induced by $\langle D \rangle$ is connected. The smallest fuzzy cardinality among all connected neighborhood sets in G is referred to as the connected neighborhood number of G, denoted as $\eta_c(G)$ or simply η_c.

The following theorem gives the relationship between $\eta_s(G)$ and $\eta_c(G)$.

Theorem 3.13: Consider D as a minimal split neighborhood set of a fuzzy graph G. If $\eta_s(G)$ is equal to $\eta_c(G)$, then it implies that V-D qualifies as a split neighborhood set of G

Proof: Since D is a minimal, then by theorem 3.13, $V - D$ is a neighborhood set of G and furthermore, it is a split neighborhood set of G, since a fuzzy subgraph induced by $\langle D \rangle$ is disconnected.

4. CONCLUSION:

In our research, we concentrated on examining the split neighborhood number in fuzzy graphs and its characterizations. As a potential avenue for future investigation, one can explore different standard graphs related to the split neighborhood number.

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