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FIXED POINT RESULT USING TWO DOMINATED MAPPING ON A CLOSED BALL

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ABSTRACT

In the present work, we obtain the result of fixed-point theorem through quasi-contraction mapping and improve the work of A. Shoaib et.al (2015) in a left and right K-sequentially 0-complete ordered quasi-partial spaces respectively, where locally contractive condition satisfied on a closed set. We can use this result to solve the complication of computer algorithms and study it.

In this paper, some fixed-point results of self-mapping which is defined on quasi partial metric spaces are given by using dominated mapping (A. Shoaib et.al, 2015) in a left and right K-sequentially 0-complete ordered quasi-partial spaces respectively. Where locally contractive condition satisfied on a closed set. And by taking advantage of these results, the necessary conditions for self-mappings on quasi partial metric spaces in quasi contraction are investigated and prove existence and uniqueness theorem of fixed point for contraction mapping. We can use this result to solve the complication of computer algorithms and study it.

Keywords: Fixed-Point Theory, Partial Metric Space, Left K-sequentially 0-complete Quasi Partial Metric Space, dominated Mappings, Cauchy Sequence, Distance Function ordered QPMS.

MSC: 47H09; 47H10; 54H25; 54C05.

1. Introduction

In various research activities, we find that in the complete domain the mapping of fixed points satisfying all the contractive conditions. These results of [8, 16, 18] are applicable in various areas such as parameterize estimation problems, fractal image decoding, nonlinear and adaptive control systems, a convergence of recurrent networks, and computing magnetostatics fields.

S. G. Mathews [25] established the Partial metric space (PMS). In PMS, self-distance may not be zero all the time. We use its application in theoretical computer science [22]. On ordered PMS, by using the concept of partial order and partial metric space, some fixed-point theorem proved by Altun et.al [15], Aydi [9], and Paesano et.al.[5] for contraction mapping.

In metric space Ran and Recrings [2] demonstrated a simple of Banach's fixed point theorem, which is presented with a partial order and gave the utilizations to matrix equation and weakened the usual contraction mappings for nondecreasing mapping. The result [2] of non-decreasing mapping expended by Nieto et.al.[17] and used it to find a unique solution. Karapinar et.al.[7] and Romaguera [26] has given the idea of quasi partial Metric Space and 0-complete partial metric space. Different authors [2-4, 10, 14, 17, 20, 21, 23, 27] did work on quasi partial metric space (QPMS) and closed ball and using the above concept they proved some valuable results.

In so many applications we find that on the whole space, the mapping is not contractive but it is contractive in its subspace. On a closed ball in a 0-complete QPMS, Shoaib et. al [1] shows that the existence of fixed points with self-mappings which are dominated and satisfied some contractive conditions. The notion of left (right) K-Cauchy sequence and left (right) K-sequentially complete spaces introduced by Reilly et.al.[13] and [4, 24].

2. Preliminaries

Definition 1.1 [7, 12]: A function $q : U \times U \rightarrow R^+$ is quasi-partial function when it satisfied

a. If $0 \le q(u,u) = q(u,v) = q(v,v)$ then $u = v$ (Equality)

b. $q(u,u) \le q(u,v)$ (Small self-distance)

c. $q(u,u) \le q(v,u)$ (Small self-distance)

d. $q(u,z) + q(v,v) \le q(u,v) + q(v,z)$ (triangle inequality)

All u, v and $z \in U$ then the pair (U, q) is called a quasi-PMS.

QPMS (U,q) becomes a PMS (U,p) if $q(u,v) = q(v,u)$. In addition, (U,q) and (U,p) known as a quasi - metric space (QMS) and a metric space if $q(u,u) = 0$. The function $d_{p_q} : U \times U \rightarrow R^+$ is a usual metric and defined by

$$d_{p_q}(u,v) = q(u,v) + q(v,u) - q(u,u) - q(v,v).$$

In QPMS, $B(u,\varepsilon)$ is an open ball defined by $B(u,\varepsilon) = \{v \in U : q(u,v) < \varepsilon + q(u,u)\}$ where $u \in U$ and $\varepsilon > 0$ and $B(u,\varepsilon)$ is a closed subset of U.

Definition 1.2 [1]: Suppose (U,q) is a QPMS then

a. $\{u_n\}$ is a sequence which is belongs to (U,q) is known as 0 – Cauchy sequence if

$$\lim_{n,m\rightarrow\infty} q(u_n, u_m) = 0 = \lim_{n,m\rightarrow\infty} q(u_m, u_n) = 0.$$

b. If $\lim_{n\to\infty} q(u_n, u) = \lim_{n\to\infty} q(u, u_n) = q(u, u) = 0$ then the sequence $\{u_n\}$ is converge to a point u.

c. In U, if every 0-Cauchy sequence converge to a point u belongs to U then (U, q) is called 0-complete space so that $q(u, u) = 0$.

In (U, q), every 0-Cauchy sequence is Cauchy in $\left(U, d_{p_q}\right)$ and (U, q) is 0-complete if it is complete but its converse in not true.

Lemma 1.2.1 [28]: In a 0-complete PMS each closed subset is 0-complete.

Definition 1.3 [28]: Let $(U, \preccurlyeq, q)$ is called an ordered QPMS, in a non-empty set U, if q is quasi-partial metric and $\preccurlyeq$ is a partial order in U.

Definition 1.4 [28]: Suppose $(U, \preccurlyeq)$ be known as partial ordered set then $u, v \in U$ are comparable which holds $u \preccurlyeq v$ or $v \preccurlyeq u$.

Definition 1.5 [28]: Suppose $(U, \preccurlyeq)$ be known as partial ordered set and f is a self-map on U. If $fu \preccurlyeq u$ for every $u \in U$ then f is called dominated mapping.

Definition 1.6 [20]:

a. For every $n > m$, if $\lim_{n,m\to\infty} q(u_m, u_n) = 0$ as well as $\lim_{n,m\to\infty} q(u_n, u_m) = 0$ then sequence $\{u_n\}$ is called left (right) K-0-cauchy in a QPMS (U, q).

b. If every left (right) K-0-cauchy sequence in U converges to a point $u \in U$ such that $q(u, u) = 0$. Then (U, q) is known as left (right) K-sequentially space.

2. Main Results:

Theorem 2.1: Let $(U, \preccurlyeq q)$ *and* $(S, \preccurlyeq q_1)$ be two left K-sequentially 0-complete ordered quasi-partial metric space. $R : U \to U$ and $T : S \to S$ be two dominated map and u_0 and s_0 be two arbitrary points in U and S.

Suppose that (a, b) and $(c, d) \in [0, 1)$ such that $(a + 2b) < 1$ and $(c + 2d) < 1$ and

$$q(Ru, Rv) + q_1(Ts, Tp) \le \{a\, q(u, v) + b[q(u, Ru) + q(v, Rv)] + c q_1(s, p) + d[q_1(s, Ts) + q_1(p, Tp)]\} \quad \text{......... (2.1.1)}$$

For all $u, v \in \overline{B(u_0, r)}$ and all $s, p \in \overline{B(s_0, r)}$, where u, v and s, p is comparable.

$$q(u_0, Ru_0) + q_1(s_0, Ts_0) \le (1 - k)[r + q(u_0, u_0)] + (1 - l)[r + q_1(s_0, s_0)] \quad \text{.........(2.1.2)}$$

Where $k = \frac{a+b}{1-b}$ and $l = \frac{c+d}{1-d}$

Then $\{u_n\} \in \overline{B(u_0, r)}$ and $\{s_n\} \in \overline{B(s_0, r)}$ have unique fixed points u^* and s^* such that $u^* = Ru^*$ with $q(u^*, u^*) = 0$ similarly $s^* = Ts^*$and $q_1(s^*, s^*) = 0$. $\{u_n\} \in \overline{B(u_0, r)}$ and $\{s_n\} \in \overline{B(s_0, r)}$ are two non-increasing sequence such that $\{u_n\} \to x$ and $\{s_n\} \to y$ which implies $x \preccurlyeq u_n$ and $y \preccurlyeq s_n$ then there exist two points $u^* \in \overline{B(u_0, r)}$ and $s^* \in \overline{B(s_0, r)}$ such that $u^* = R^*$ and $q(u^*, u^*) = 0$ similarly $s^* = Ts^*$ and $q_1(s^*, s^*) = 0$.

Proof: Here we consider two Picard sequences $u_{n+1} = Ru_n$ and $s_{n+1}= Ts_n$ with initials u_0 and s_0 (suppose). As $u_{n+1} = Ru_n \preccurlyeq u_n$ and $s_{n+1} = Ts_n \preccurlyeq s_n$ for all $n \in \{0\} \cup N$.

Now we will prove by using mathematical induction $u_n \in \overline{B(u_0, r)}$ and $s_n \in \overline{B(s_0, r)}$

By using the inequality (2.1.2),

$$q(u_0 , u_1) + q_1 (s_0 , s_1) \leq (1 - k) [r + q(u_0 , u_1)] + (1 - l) [r + q_1 (s_0 , s_1)]$$
$$\leq [r + q (u_0 , u_0)] + [r +q_1 (s_0, s_0)]$$

Therefore $u_1 \in \overline{B(u_0, r)}$ and $s_1 \in \overline{B(s_0, r)}$. Now let $u_2, u_3 ..... u_i \in \overline{B(u_0, r)}$ and $s_2, s_3 ..... s_j \in \overline{B(s_0, r)}$.

For some $i, j \in N$. As $u_{n+1} \preccurlyeq u_n$ and $s_{n+1} \preccurlyeq s_n$. By using inequality (2.1.1), we get,

$$q(u_i , u_{i+1}) + q_1(s_j , s_{j+1}) = q(Ru_{i-1}, Ru_i) + q_1(Ts_{j-1} , Ts_j)$$
$$\leq a[q(u_{i-1} , u_i)] + b[q(u_{i-1} , u_i) + q(u_i , u_{i+1})] + c[q_1(s_{j-1} , s_j)] + d[q_1(s_{j-1} , s_j) + [q_1(s_j , s_{j+1})]$$
$$q(u_i , u_{i+1}) - bq(u_i , u_{i+1},) + q_1(s_j , s_{j+1}) - dq_1(s_j , s_{j+1})]$$
$$\leq a[q(u_{i-1} , u_i)] + b[q(u_{i-1} , u_i)] + c[q_1(s_{j-1} , s_j) + d[q_1(s_{j-1} , s_j)]$$

Now we separate the equations

$\Rightarrow (1 - b)q(u_i , u_{i+1})\leq (a + b) [q(u_{i-1} , u_i)]$ and $(1 - d) q_1(s_j , s_{j+1}) \leq (c + d) [q_1(s_{j-1} , s_j)]$

$\Rightarrow q(u_i , u_{i+1})\leq \left\{\frac{a+b}{1-b}\right\} [q(u_{i-1} , u_i)]$ and $[q_1(s_j , s_{j+1}) \leq \left\{\frac{c+d}{1-d}\right\} [q_1(s_{j-1} , s_j)]$

$\Rightarrow q(u_i , u_{i+1}) \leq k [q(u_{i-1} , u_i)]$ and $q_1(s_j , s_{j+1}) \leq l[q_1(s_{j-1} , s_j)]$

which implicit that $q(u_i , u_{i+1}) \leq k^2[q(u_{i-2} , u_{i+1})] \leq .... \leq k^i[q(u_0 , u_1)]$

and $\{q_1(s_j , s_{j+1}) \leq l^2[q_1((s_{j-2} , s_{j-1})] \leq .... \leq l^j[q_1(s_0 , s_1)]\}$ ....(2.1.3)

Now $q (u_0, u_{i+1}) + q_1(s_0 , s_{j+1}) \leq \{q(u_0, u_1) + ... + q(u_i , u_{i+1}) - [q(u_1 , u_1) + ... + q(u_i ,u_i)]\}$

$+ \{q_1(s_0 , s_1) + ... + q_1(s_j, s_{j+1}) - [q_1(s_1, s_1) + ... + q_1(s_j, s_j)]\}$

$\leq \{q_1(u_0 , u_1)[1 + ... + k^{i-1} + k^i] + q_1(s_0, s_1)[1 + ... + l^{j-1}+l^j]$- - - - - by (2.1.3)

$\leq (1 - k) [r + q(u_0 , u_0)] \frac{1-k^{i+1}}{1-k}+ (1 - l) [r + q_1(s_0 , s_0)] \frac{1-l^{j+1}}{1-l}$- - - - - by (2.1.2)

Thus $u_{i+1} \in \overline{B(u_0 , r)}$ and $s_{j+1} \in \overline{B(s_0 , r)}$. Hence $u_n \in \overline{B(u_0 , r)}$ and $s_n \in \overline{B(s_0 , r)}$ for all $n \in N$.

(i) Now we prove $q(u^*, u^*) = 0$ and $q_1(s^*, s^*) = 0$

Since u_{i+1} and $u_n \in \overline{B(u_0 , r)}$ similarly s_{j+1} and $s_n \in \overline{B(s_0 , r)}$.

It also shows that $u_{n+1} \preccurlyeq u_n$ and $s_{n+1} \preccurlyeq s_n$ for all $n \in N$ which implicit that

$q(u_n , u_{n+1}) + q_1(s_n , s_{n+1}) \leq k^n[q(u_0 , u_1)] + l^n[q_1(s_0 , s_1)]$ for all $n \in N$

It follows that $q(u_n , u_{n+e}) + q_1(s_n , s_{n+f}) \leq$

$\{[q(u_n , u_{n+1}) + \ldots\ldots\ldots + q(u_{n+e-1} , u_{n+e})] - q(u_{n+1} , u_{n+1}) + \ldots\ldots\ldots + q(u_{n+e-1} , u_{n+e-1})\}$

$+ \{[q_1(s_n , s_{n+1}) + \ldots. + q_1(s_{n+f-1}, S_{n+f})] - q_1(S_{n+1}, S_{n+1}) + \ldots\ldots\ldots + q_1(s_{n+f-1} , s_{n+f-1})\}$

$q(u_n , u_{n+e}) + q_1(s_n , s_{n+f}) \leq \{ k^n q(u_0 , u_1)[1 + \ldots\ldots + k^{e-2} + k^{e-1}] +$

$l^n q_1(s_0, s_1) [1 + \ldots\ldots + l^{f-2} + l^{f-1}]\} \to 0$ as $n \to \infty$............................(a)

The sequence $\{u_n\} \in \overline{B(u_0 , r)}$ and $\{s_n\} \in \overline{B(s_0 , r)}$ are two left k-0 Cauchy sequences. As $\overline{B(u_0 , r)}$ and $\overline{B(s_0 , r)}$ are closed. So, it is left k-sequentially 0-complete.

Then there exist two-point $u^* \in \overline{B(u_0 , r)}$ and $s^* \in \overline{B(s_0 , r)}$ with

$\{q(u^*, u^*) = \lim_{n\to\infty} q(u_n , u^*) = \lim_{n\to\infty} q(u^*, u_n) = 0$ and

$q_1(s^*, s^*) = \lim_{n\to\infty} q_1 (s_n , s^*) = \lim_{n\to\infty} q_1 (s^*, s_n) = 0\}$...........(2.1.4)

From inequality (2.1.4) $q(u^*, u^*) = 0 = q_1(s^*, s^*)$

(ii) Prove that $u^* = Ru^*$ and $s^* = Ts^*$.

Now $q(u^*, Ru^*) + q_1(s^*, Ts^*) \leq [q(u^*, u_n) + q(Ru_{n-1}, Ru^*) - q(u_n , u_n)]+$

$[q_1(s^*, s_n) + q_1(Ts_{n-1} , Ts^*) - q_1(s_n , s_n)]$

While taking limit as $n \to \infty$ and using also $u^* \preccurlyeq u_n \preccurlyeq u_{n-1}$ and $s^* \preccurlyeq s_n \preccurlyeq s_{n-1}$,

When $u_n \to u^*$ and $s_n \to s^*$, we have

$q(u^*, Ru^*) + q_1(s^*, Ts^*) \leq [\lim_{n\to\infty} q(u^*, u_n) + a\, q(u_{n-1} , u^*) + b\{q(u_{n-1} , Ru_{n-1}) + q(u^*, Ru^*)\}]$

$+ [\lim_{n\to\infty} q_1(s^*, s_n) + c q_1(s_{n-1}, s^*) + d\{q_1(s_{n-1}, Ts_{n-1}) + q_1(s^*, Ts^*)\}]$

$q(u^*,Ru^*) + q_1(, s^*, Ts^*) \leq [\lim_{n\to\infty} q(u^*, u_n) + a\, q(u_{n-1}, u^*) + b\{ k^{n-1} q(u_0 , u_1) + q(u^* , Ru^*)\}]$

$+ [\lim_{n\to\infty} q_1(s^*, s_n) + c q_1(s_{n-1}, s^*) + d\{l^{n-1} q_1(s_0, s_1) + q_1(s^*, Ts^*)\}]$

$(1 - b)\, q(u^*, Ru^*) + (1 - d) q_1(s^*, Ts^*) \leq [\lim_{n\to\infty} q(u^*, u_n) + a\, q(u_{n-1}, u^*) + b\{q(u_{n-1},Ru_{n-1})\}]$

$+ [\lim_{n\to\infty} q_1(s^*, s_n) + c q_1(s_{n-1}, s^*) + d\{q_1(s_{n-1}, Ts_{n-1})\}]$

From (2.1.4) and (a) we get

$(1 - b)\, q(u^*,Ru^*) + (1 - d) q_1(s^*, Ts^*) \leq 0$

$q(u^*,Ru^*) + q_1(s^*, Ts^*) \leq 0$

so $q(u^*,Ru^*) \leq 0$ and $q_1(s^*, Ts^*) \leq 0$

Hence $u^* = Ru^*$ and $s^* = Ts^*$

UNIQUENESS:

Now we have to prove, u^* and s^* are unique, if any two points $u, v \in \overline{B(u_0 , r)}$ and two points $s, p \in \overline{B(s_0 , r)}$ there exists a point $w \in \overline{B(u_0 , r)}$ and $z \in \overline{B(s_0 , r)}$ such that $w \preccurlyeq u$ and v as well as $z \preccurlyeq s$ and p and

$[q(u_0 , Ru_0) + q(w , Rw)] + [q_1(s_0 , Ts_0) + q_1(z , Tz)] \leq [q(u_0 ,w) + q(Ru_0 , Rw)]+$

$+ [q_1(s_0, z) + q(Ts_0, Tz)]$ Where all $w \preccurlyeq Ru_0$ and $z \preccurlyeq Ts_0$ ……………..(2.1.5)

Let $v \in \overline{B(u_0, r)}$ and $p \in \overline{B(s_0, r)}$ such that v = Rv and p = Tp. Then

$q(v, v) + q_1(p, p) = q(Rv, Rv) + q_1(Tp, Tp)$

$q(v, v) + q_1(p, p) \leq aq(v, v) + b\{q(v, Rv) + q(v, Rv)\} + cq_1(p, p) + d\{q_1(p, Tp) + q_1(p, Tp)\}$

$q(v, v) + q_1(p, p) - aq(v, v) - b\{q(v, Rv) + q(v, Rv)\} - cq_1(p, p) - d\{q_1(p, Tp) + q_1(p, Tp)\} \leq 0$

$q(v, v) - aq(v, v) - 2bq(v, Rv) + q_1(p, p) - cq_1(p, p) - 2dq_1(p, Tp) \leq 0$

$q(v, v) - a\,q(v, v) - 2bq(v, v) + q_1(p, p) - cq_1(p, p) - 2dq_1(p, p) \leq 0$ $\{v = Rv$ and $p = Tp\}$

$(1 - a - 2b)\, q(v, v) + (1 - c - 2d)\, q_1(p, p) \leq 0$

Hence $q(v, v)$ and $q_1(p, p) = 0$ …………… (2.1.6)

If $u^* \preccurlyeq v$ and $s^* \preccurlyeq p$ than

$q(u^*, v) + q_1(s^*, p) = q(Ru^*, Rv) + q_1(Ts^*, Tp)$

$\leq aq(u^*, v) + b\{q(u^*, Ru^*) + q(v, Rv)\} + cq_1(s^*, p) + d\{q_1(s^*, Ts^*) + q_1(p, Tp)\})$

$q(u^*, v) - aq(u^*, v) + q_1(s^*, p) - cq_1(s^*, p) \leq b\{q(u^*, Ru^*) + q(v, Rv)\} + d\{q_1(s^*, Ts^*) + q_1(p, Tp)\}$

$(1 - a)\, q(u^*, v) + (1 - c)\, q_1(s^*, p) \leq 0$- - - - - - - By(2.1.4) and (2.1.6)

Therefore $q(u^*, v) \leq 0$ and $q_1(s^*, p) \leq 0$ similarly we can show that $q(v, u^*) \leq 0$ and $q_1(p, s^*) \leq 0$.

This shows that $u^* = v$ and $s^* = p$.

Now, if u^* and v and as well as s^* and p are not comparable then there exists two points $w \in U$ and $z \in S$ where w is lower bound of both u^* and v that is $w \preccurlyeq u^*$ and $w \preccurlyeq v$ similarly z is lower bound of both s^* and p that is $z \preccurlyeq s^*$ and $z \preccurlyeq p$.

(iii) Now we will prove here $R_n w \in \overline{B(u_0, r)}$ and $T^n z \in \overline{B(s_0, r)}$

By using inequality (2.1.1) and mathematical induction, Suppose $w \preccurlyeq u^* \preccurlyeq u_n \ldots\ldots \preccurlyeq u_0$ and

$z \preccurlyeq s^* \preccurlyeq s_n \ldots\ldots \preccurlyeq s_0$

Then $q(Ru_0, Rw) + q_1(Ts_0, Tz) \leq aq(u_0, w) + b\{q(u_0, u_1) + q(w, Rw)\} + cq_1(s_0, z) + d\{q_1(s_0, s_1) + q_1(z, Tz)\}$

$\leq a\, q(u_0, w) + b\{q(u_0, w) + q(u_1, Rw)\} + cq_1(s_0, z) + d\{q_1(s_0, z) + q_1(s_1, Tz)\}$ – – – – – by(2.1.3)

So $q(u_1, Rw) - b\, q(u_1, Rw) \leq a\, q(u_0, w) + b\, q(u_0, w)$

and $q_1(s_1, Tz) - d\, q_1(s_1, Tz) \leq cq_1(s_0, z) + dq_1(s_0, z)$

$(1 - b)\, q(u_1, Rw) \leq (a + b)q(u_0, w)$ and $(1 - d)q_1(s_1, Tz) \leq (c + d)q_1(s_0, z)$

$q(u_1, Rw) \leq \left\{\frac{a+b}{1-b}\right\} q(u_0, w)$ and $q_1(s_1, Tz)$ $\left\{\frac{c+d}{1-d}\right\} q_1(s_0, z)$

$q(u_1, Rw) \leq k\, q(u_0, w)$ and $q_1(s_1, Tz) \leq l\, q_1(s_0, z)$

Now, $q(u_1, Rw) + q_1(s_1, Tz) \leq k\, q(u_0, w) + l\, q_1(s_0, z)$ ……………….. (2.1.7)

Now $q(u_0, Rw) + q_1(s_0, Tz) \leq [q(u_0, u_1) + q(u_1, Rw) - q(u_1, u_1)] + [q_1(s_0, s_1) + q_1(s_1, Tz) - q_1(s_1, s_1)]$

$q(u_0, Rw) + q_1(s_0, Tz) \le q(u_0, u_1) + k\, q(u_0, w) + q_1(s_0, s_1) + l\, q_1(s_0, z)$- - - - - - -by (2.1.7)

$\le \{1 - k)\,[r + q(u_0, u_0)] + k\,[r + q(u_0, u_0)]\}+\{1 - l)\,[r + q_1(s_0, s_0)] + l\,[r + q_1(s_0, s_0)]\} = r$

It follows that $Rw \in \overline{B(u_0, r)}$ and $Tz \in \overline{B(s_0, r)}$. Let $R^2w, R^3w,$ ....$R^e w \in \overline{B(u_0, r)}$ and,

$T^2z, T^3z, \ldots T^f z \in \overline{B(s_0, r)}$.

for all e and f ∈ N. As $R^e w \preccurlyeq R^{e-1} w \preccurlyeq \ldots \preccurlyeq w \preccurlyeq u^* \preccurlyeq u_n \ldots \preccurlyeq u_0$

and $T^f z \preccurlyeq T^{f-1} z \preccurlyeq \ldots \preccurlyeq z \preccurlyeq s^* \preccurlyeq s_n \ldots \preccurlyeq s_0$,

Then $q(u_1, R^{e+1}w) + q_1(s_1, T^{f+1}z) = [a\, q(u_0, R^e w) + b\{q(u_0, u_1) + q(R^e w, R^{e+1}w)\}]+$

$+ [cq_1(s_0, T^f z) + d\{q_1(s_0, s_1) + q_1(T^f z, T^{f+1}z)\}]$

$\le [a\, q(u_0, R^e w) + b\{q(u_0, R^e w) + q(u_1, R^{e+1}w\}] + [cq_1(s_0, T^f z)$

$+d\{q_1(s_0, T^f z)+q_1(s_1, T^{f+1}z)\}]$ – – – by (2.1.5)

$q(u_1, R^{e+1}w) - b\, q(u_1, R^{e+1}w) + q_1(s_1, T^{f+1}z) - dq_1(s_1, T^{f+1}z)]$

$\le [a\, q(u_0, R^e w) + b\{q(w_0, R^e w)] + [cq_1(s_0, T^f z) + d\{q_1(s_0, T^f z)]$

$(1 - b)\, q(u_1, R^{e+1}w) \le (a + b)q(u_0, R^e w)$ and $(1 - d)q(s_1, T^{f+1}z) \le (c + d)q_1(s_0, T^f z)$

$q(u_1, R^{e+1}w) \le \left\{\frac{a+b}{1-b}\right\} q(u_0, R^e w)$ and $q_1(s_1, T^{f+1}z) \le \left\{\frac{c+d}{1-d}\right\} + q_1(s_0, T^f z)$

Which implies that

$q(u_1, R^{e+1}w) + q_1(s_1, T^{f+1}z) \le k\, q(u_0, R^e w) + l\, q_1(s_0, T^f z)$

$\le k\,[r + q(u_0, u_0)] + l\,[r + q_1(s_0, s_0)]$ as $R^e w \in \overline{B(u_0, r)}$ and $T^f z \in \overline{B(s_0, r)}$. …………..(2.1.8)

$q(u_0, R^{e+1}w) + q_1(s_0, T^{f+1}z) \le [q(u_0, u_1) + q(u_1, R^{e+1}w) + [q_1(s_0, s_1) + q_1(s_1, T^{f+1}z)]$

$\le \{(1 - k)\,[r + q(u_0, u_0)] + k\,[r + q(u_0, u_0)]\} + \{(1 - l)\,[r + q_1(s_0, s_0)] + l\,[r + q_1(s_0, s_0)]\} = r$

It shows that $R^{e+1} \in \overline{B(u_0, r)}$ and $T^{f+1}z \in \overline{B(s_0, r)}$

Hence $R^n w \in \overline{B(u_0, r)}$ and $T^n z \in \overline{B(s_0, r)}$.

As $R^n w \preccurlyeq wR^{n-1}w \preccurlyeq \ldots \preccurlyeq w$ and $T^n z \preccurlyeq T^{n-1}z \preccurlyeq \ldots \preccurlyeq z$ ………………….(2.1.9)

(iv) Now we prove u^* and s^* are unique

From (2.1.9)

As $q(R^n w, R^{n+1}w) + q_1(T^n z, T^{n+1}z) \le \{aq(R^{n-1}w, R^n w) + b[q(R^{n-1}w, R^n w)$

$+q(R^n w, R^{n+1}w)]\} + \{cq_1(T^{n-1}z, T^n z) + d[q_1(T^{n-1}z, T^n z) + q_1(T^n z, T^{n+1}z)]\}$

Which implies that

$q(R^n w, R^{n+1}w) + q_1(T^n z, T^{n+1}z) \le k\, q(R^{n-1}w, R^n w) + l\, q_1 + (T^{n-1}z, T^n z)$

$\le [k^2 q(R^{n-2}w, R^{n-1}w) \le \ldots \le k^n q(w, Rw) + l^2 q_1(T_{n-2}z, T_{n-1}z) \le \ldots \le l^n q_1(z, Tz)] \to 0$

as $n \to \infty$ …………..(2.1.10)

Now $q(u^*, v) + q_1(s^*, p) = q(Ru^*, Rv) + q_1(Ts^*, Tp)$

$\le [q(Ru^*, R^{n+1}w) + q(R^{n+1}w, Rv) + q(R^{n+1}w, R^{n+1}w)]$

$+ [q_1(Ts^*, T^{n+1}z) + q_1(T^{n+1}z, Tz) + q_1(T^{n+1}z, T^{n+1}z)]$

As $R^{n-1}w \preccurlyeq u^*$ and $R^{n-1}w \preccurlyeq v$ similarly $T^{n-1}z \preccurlyeq s^*$ and $T^{n-1}z \preccurlyeq p$ for all $n \in N$

Which shows that $R^{n-1}w \preccurlyeq R^n u^*$ and $R^{n-1}w \preccurlyeq R^n v$ for all $n \in N$ as $R^n u^* = u^*$ and $R^n v = v$.

Similarly, $T^{n-1}z \preccurlyeq T^n s^*$ and $T^{n-1}z \preccurlyeq T^n p$ for all $n \in N$ as $T^n s^* = s^*$ and $T^n p = p$ for $n \in N$. Then

$$q(u^*, v) + q_1(s^*, p) \leq [a\, q(u^*, R^n w) + b\{q(u^*, Ru^*) + q(R^n w, R^{n+1} w)\} + aq(R^n w, v) + b\{qR^n w, R^{n+1} w) + q(v, Rv)\}] + [cq_1(s^*, T^n z) + d\{q_1(s^*, Ts^*) + q_1(T^n z, T^{n+1} z)\} + cq_1(T^n z, p) + d\{q_1(T^n z, T^{n+1} z) + q_1(p, Tp)\}]$$

Now using inequalities (2.1.6), (2.1.10) and taking limit as $n \to \infty$ then we have

$$q(u^*, v) + q_1(s^*, p) \leq \lim_{n\to\infty} [aq(u^*, R^n w) + aq(R^n w, v) + cq_1(s^*, T^n z) + cq_1(T^n z, p)]$$

$$\leq \lim_{n\to\infty} [a^2 q(u^*, R^{n-1} w) + a^2 q(R^{n-1} w, v)] + c^2 q_1(s^*, T^{n-1} z) + c^2 q_1(T^{n-1} z, p)]$$

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$$\leq \lim_{n\to\infty} [a^n q(u^*, Rw) + a^n q(Rw, v)] + [c^n q_1(s^*, Tz) + c^n q_1(Tz, p)] \to 0$$

So $q\,[(u^*, v) + q_1(s^*, p) \leq 0$ or $q\,(u^*, v) \leq 0$ and $q_1(s^*, p) \leq 0$ ……………..(2.1.11)

Similarly, $q\,[(v, u^*) + q_1(p, s^*) \leq 0$ or $q(v, u^*) \leq 0$ and $q_1(p, s^*) \leq 0$ ……..(2.1.12)

From inequalities (1.11) and (1.12) we find $u^* = v$ and $s^* = p$.

It shows that u^* and s^* are unique in their respective ball.

Conclusion:

In this chapter, we proposed and improved the fixed-point results to generalized and extended the work of literature [28] by using two dominated mapping with two different ball conditions. This work generalized the work of [3, 4, 20, 21, 23] with the help of closed ball and using weaker contractive conditions method. Finally, the existence and uniqueness of unique fixed points in two QPMS has been achieved.

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