Leaf Domination of Graphs

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ABSTRACT

Let G be connected graph. A dominating set D in G is said to be a leaf dominating set if $< D >$ must have at least one leaf vertex. The least number of a leaf dominating set is said to be leaf domination number, represented by $\gamma_{ld}(G)$. Any minimal dominating set with the least amount of cardinality is called the $\gamma_{ld} -$ set. In this work, we determine leaf domination number and calculate leaf domination number for standard graphs.

Keywords: Domination Number, Isolate Domination, Wheel graph, Ladder graph, Pan graph.

1. Introduction

Numerous graph theory researchers have been drawn to the idea of domination in graphs, and throughout the past few decades, numerous domination parameters—such as mixed, connected, strong, and total—have been developed and investigated. The idea of domination can be used to determine the minimum number of security personnel, with one guard assigned as a backup to the other. The essential idea of isolate domination in graphs was recently established by I.S. Hamid [7]. To learn more about the isolate domination number, see [1, 2, 11, 12, 13]. In this research, we study and offer a new domination parameter called leaf domination, for which a backup guard is given to at least one guard, inspired by the isolated domination notion. If there is at least one isolated vertex in the subgraph that S induces, then the subset S of vertices is considered an isolated dominating set. The isolated dominating number, represented as, is the lowest number of an isolated dominating set. The highest isolated domination number, indicated by, is the minimal isolated dominating set of maximal cardinality. If each and every vertex v in subset of vertices, let's say S, has at least one private neighbour $pn[v, s] \neq \emptyset$, then that subset is said to be irredundant.

Therefore, the concept of an irredundant set serves as the minimality criterion for a dominant set. The irredundance number, $(ir(G))$, is the cardinality bare minimum of a maximal irredundant collection. A maximal irredundant set's maximum cardinality is known as its higher irredundance number, and it is represented by IR(G). In the beginning, Cockayne, Hedetniemi, and Miller defined these concepts.

We recall the following results Theorem[6] dominating set S is called minimal dominating set iff for every vertex, One of the subsequent conditions applies:.

1. u must be an isolate of S.

2. there exists vertex $u \in V - S$ for whatever $N(v) \cap S = \{u\}$

Theorem [6] For any graph G, $\gamma(G) = i(G)$ if G must be a claw free graph.

Theorem [10] If a graph G has no isolated vertices then $\gamma(G) \leq \frac{n}{2}$.

2. Basic Definitions and Notations

This section is devoted to recall the definitions and notations that are used in the present paper. In fact, at this point, we assume a familiarity with fundamental definitions of basic graph theory, and we refer the reader to the books [3, 5].Definitions that are repeatedly used are defined here and that are less common are defined additionally at the point at which the term is discussed.

A graph $G = (V, E)$ consists of $V = V(G)$, a non-empty collection of objects known as nodes or vertices, and $E = E(G)$, an unordered pair of different nodes of G known as edges. A graph's order is the number of vertices, and its size is the number of edges.

For a vertex v in a graph G, the count of vertices adjacent to v, is called the degree of vertex v, is called degree of vertex v, denoted by $\deg(v)$. The degree of an edge $e = uv$ is defined as

$$\deg(e) = \deg(u) + \deg(v) - 2.$$

A vertex v is mentioned an isolated vertex if $\deg(v) = 0$ and is mentioned pendant vertex if the $\deg(v)$ is one. Any vertex adjacent to a pendant vertex is mentioned support vertex.

Further, $\delta = \delta(G)$ represents min. degree of vertex in G and $\Delta = \Delta(G)$ denotes maximum degree of a vertex in G.

For any node v in $V(G)$, the open neighborhood of v is denoted by $N(v)$ and is defined by

$$N(v) = \{u \in V / uv \in E\},$$

the collection of all Vertices are nearby to v. The closed neighborhood of v is denoted $N[v]$ and defined by $N[v] = N(v) \cup \{v\}$.

For any subset S of G, the open and closed neighborhood of S in G is defined by $N(S) = \bigcup_{v \in S} N[v]$ respectively.

Let S be subset of vertices and $u \in S$ be arbitrary. A vertex v is considered a private neighbor of u with respect to S if if $N[v] \cap S = \{u\}$. Furthermore, The open set that defines the private neighbor set of u, in relation to S, is formally defined $pn[u, S] = \{v / N[v] \cap S = \{u\}\}$. It can be observed from the definition $u \in pn[u, S]$ if u would be an isolated vertex on $< S >$, in that case we could that u is private neighbor for its own.

A graph G is called to be connected if for each couple of vertices are linked by a path. If each couple of vertices are adjacent, then G is called complete graph. The maximal connected sub graph is referred as a component of G. Hence; the graph G is disconnected if it has at least two components.

A graph G is labeled a bipartite graph if it is possible to split the vertex set $V(G)$ in to two sets V_1 and V_2 in order that no two vertices from the same set are adjacent, and node in V_1 is adjacent to a vertex in V_2 and vice-versa.

If each vertex in V_1 is adjacent to each vertex in V_2, then G is called complete bipartite graph. If G is labeled complete bipartite graph with $|V_1| = m$ and $|V_2| = n$, then G is denoted by $K_{m,n}$. A complete bipartite graph $K_{1,n}$ is called to be star. If $V(G)$ can be partitioned into $\{V_1, V_2, \dots, V_n\}$ such that any edge in G has one end in V_1 and another end in $V_j, i \neq j$. Then G is called multi-partite graph. A complete tri-partite graph $K_{1,1,2}$ is called a diamond graph. A graph not containing a sub graph isomorphic to $K_{1,3}$ is called as claw-free graph.

A tadpole graph or $(m, n) -$ tadpole graph is mentioned to be graph of order $m + n$ attained by the adjoining path P_m to a cycle C_n with a bridge. If $m = 1$, then it is called a pan graph. A Barbell Graph is attained by joining two duplicates of a complete Graph with bridge.

A graph G is called a bistar (sometimes called, double star) if it can be constructed from K_2 by touching m edges in one vertex and n in the another vertex, denoted by $B(m, n)$.

The multi-star graph $K_m(a_1, a_2, \dots, a_m)$ is a graph of order $a_1 + a_2 + \dots + a_m + m$ formed by joining $a_1, a_2, \dots, a_m$ end-edges to m vertices of K_m. For example, $K_2(a_1, a_2)$ is a double star.

The join $G = G_1 \lor G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2.

The corona product $G_1 \circ G_2$ is defined as the graph G obtained by taking one copy of G_1 of order n_1 and n_1 copies of G_2 and then joining the ith node of G_1 to every node in the ith copy of G_2. In particular, the graph $C_n \circ K_1$ is referred as a Sun let graph.

A crown graph [4] is an undirected graph consisting of $2n$ vertices, divided into two sets of vertices u_i and v_i. In this graph, there is an edge from u_i to v_i whenever $i \neq j$. Essentially, the crown graph can be visualized as a complete bipartite graph with the edges of a perfect matching removed.

The Cartesian product of graphs G and H is a graph $G \square H$ whose vertex set is $V(G) \times V(H)$ and their edge set is the collection of all couples $(u_1, v_1)(u_2, v_2)$ so that $u_1 u_2 \in E(G)$ and $v_1 = v_2$ or $v_1 v_2 \in E(H)$ and $u_1 = u_2$. Thus, for every edge $u_1 u_2$ of G and every edge $v_1 v_2$ of H, therein are four edges in $G \square H$, $(u_1, v_1), (u_2, v_1), (u_1, v_2), (u_1, v_1), (u_1, v_2)$ and $(u_2, v_1), (u_2, v_2)$.

The graph Cartesian product of paths P_m and P_n is called the $m \times n$ grid graph. In particular, if $m = 2$, it is referred as a ladder graph.

As usual given a real number $x, \lfloor x \rfloor$ represents highest integer less than x and $\lceil x \rceil$ represents the least integer $> x$.

A set $S \subseteq V$ is called a dominating set of G, if each vertex in V is whichever a vertex in S adjacent to a vertex in set S or in close proximity to a vertex in S. i.e., S is a dominating set of G for $N[S] = V(G)$. The small number of a dominating set in G is mentioned the domination number of G and is usually labelled by $\gamma(G)$. A $\gamma -$ set in G is a dominating set in G with the cardinality $\gamma(G)$.

3. Main Results:

Definition: A dominating set D in G is said to be a leaf dominating set if $<D>$ must have at least one leaf vertex. The least number of a leaf dominating set is said to be leaf domination number, represented by $\gamma_{ld}(G)$. Any minimal dominating set with small cardinality is called the $\gamma_{ld}-$ set.

Example 3.1:

1. For every graph G of order m with maximum degree $m-1$ then, $\gamma_{ld}(G) = 2$.

2. For a complete multi-partite graph $K_{s_1,s_2,s_3,...,s_t}$, then the number of $\gamma_{ld}(G)$ must be two.

3. Let G be a multi-star graph $K_t(b_1, b_2, ..., b_m)$. Then $\gamma_{ld}(G) = \begin{cases} 2 & if\ m = 2 \\ 3 & if\ b_j = 1\ for\ some\ j\ . \\ m+1 & otherwise \end{cases}$

Observation 3.2:

The parameter $\gamma_{ld}(G)$ is not specified for a totally disconnected graph. Therefore, we consider the graph contains minimum one edge.

Observation 3.3:

If there will be a $\gamma -$ set D of G such that $<D>$ has an isolated vertex, then

$\gamma_{ld}(G) = \gamma(G)\ or\ \gamma(G) + 1.$

Theorem 3.4: If G is a cycle or a path with t vertices. Then $\gamma_{ld}(G) = \begin{cases} \frac{t}{3}+1 & if\ t \equiv 0(mod\ 3) \\ \left\lceil \frac{t}{3} \right\rceil & if\ t \equiv 1(mod\ 3) \\ \left\lceil \frac{t}{3} \right\rceil + 1 & if\ t \equiv 2(mod\ 3) \end{cases}$

Proof: Let $G \cong P_t$ be a path and let $V(G) = \{w_1, w_2, ..., w_t\}$. We examine the following possible cases here.

Case (i) Suppose $t \equiv 0(mod\ 3)$. Then $t = 3m$ for some integer $m > 0$. Then the set

$D = \{w_1, w_{3j-1}/1 \le j \le m\}$ is leaf dominating set of G.

Hence, $\gamma_{ld}(G) \le |D|$. i.e., $\gamma_{ld}(G) \le \frac{t}{3} + 1$.

However, on the flip side, we have $\gamma_{ld}(G) = \frac{t}{3}$, and any minimum dominating set of G contains only isolated vertices. Hence, $\gamma_{ld}(G) \ge \frac{t}{3} + 1$ and so we get,

$\gamma_{ld}(G) = \frac{t}{3} + 1.$

Case (ii) Suppose $t \equiv 1(mod\ 3)$. Then, it is easy to check that any $\gamma -$ set in G contains a leaf vertex. Hence, the $\gamma - set$ in G itself a leaf dominating set in G. Therefore

$\gamma_{ld}(G) = \gamma(G) = \left\lceil \frac{t}{3} \right\rceil.$

Case (iii) Proof of this is similar to case 1.

Theorem 3.5: Let G be a Barbell graph. Then $\gamma_{ld}(G) = 2$.

Theorem 3.6: For a Pan graph G. So, $\gamma_{ld}(G) = 2 + \left\lfloor \frac{m-1}{2} \right\rfloor$.

Theorem 3.7: If G is to be a disconnected graph along with components $G_1, G_2, \ldots, G_k$. Then

$$\gamma_{ld}(G) = \min_{1 \le j \le k}\{\gamma(G_j) + \sum_{i=1, j \ne i}^{k} \gamma(G_i)\}.$$

Proof: We prove this result by using mathematical induction. Since G is disconnected, $k \ge 2$. Suppose $k = 2$. Then $G = G_1 \cup G_2$. Let S_1, S_2 be the γ_{ld}- sets of G_1 and G_2 respectively. Then $S_1 \cup S_2'$ and $S_2 \cup S_1'$ are pendant dominating sets in G, where S_j' denotes the $\gamma -$ set of G_j, $j = 1,2$. Therefore,

$$\gamma_{ld}(G) \le \min\{\gamma_{ld}(G_1) + \gamma(G), \gamma_{ld}(G_2) + \gamma(G_1)\}.$$

On the other hand, Let D be any leaf dominating set in G. Then D has to dominate both $V(G_1)$ and $V(G_2)$ and $< S >$ should contain at least one leaf vertex. Moreover, the set D should contain the leaf dominating set of G_1 or G_2. Otherwise $< D >$ contains no leaf vertex which is a contradiction. This contradiction shows that $|D| \ge \min\{\gamma_{ld}(G_1) + \gamma(G_2), \gamma_{ld}(G_2) + \gamma(G_1)\}$ proving $k = 2$.

Next, Suppose $k \ge 3$ and assume that the result is true for $k = t - 1$. Let G be any graph with the components $G_1, G_2, \ldots, G_{t-1}, G_t$.

Let G' be a graph with $t - 1$ components, say $G_1, G_2, \ldots, G_{t-1}$. Then from the induction hypothesis we have

$$\gamma_{ld}(G') = \min_{1 \le \lambda \le t-1}\{\gamma_{ld}(G_\lambda) + \sum_{j=1, j \ne \lambda}^{t-1} \gamma(G_\lambda)\}.$$

Now, We have $G = G' \cup G_k$. Now, from the case $k = 2$, we obtain that

$$\gamma_{ld}(G) = \min_{1 \le \lambda \le t}\{\gamma_{ld}(G_\lambda) + \sum_{j=1, j \ne \lambda}^{t} \gamma(G_j)\}.$$

Therefore, the result is true for $k = t$ and hence true for any positive integer k. Thus we have $\gamma_{ld}(G) = \min_{1 \le \lambda \le k}\{\gamma_{ld}(G_\lambda) + \sum_{j=1, j \ne \lambda}^{k} \gamma(G_j)\}$.

Theorem 3.8: Let G_1 and G_2 be any two graphs. Then $\gamma_{ld}(G_1 \vee G_2) = 2$.

Proof: As a consequence of the above theorem, the value of $\gamma_{ld}(G)$ is 2 if G is a wheel, Fan graph, or a Crown graph. In the following theorem, we determine $\gamma_{ld}(G)$ for corona of two graphs.

Theorem 3.9: Let G be a connected graph with m vertices and M be any graph. Then,

$$\gamma_{ld}(G \circ M) = \begin{cases} m + 1 & \text{if } G \text{ is a cycle and } \gamma(M) \ge 2 \\ m & \text{otherwise} \end{cases}.$$

4. Conclusion

In this wok, we determined leaf domination number and found leaf domination number of standard graphs .

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