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BAYESIAN STUDY OF NORMAL SEQUENCE APPLYING A PRIOR MIXTURE TO THE LOCATION PARAMETER

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ABSTRACT

This article discusses Bayesian Analysis of Normal Sequence using Mixture of Priors. Using the Bayesian methodology, one can generate Bayes estimates by assuming a newly created Mixture combination of priors for location and correct prior for scale parameters. In order to get Bayes estimates for our study, we have assumed that one innovative form of prior, such as a double exponential prior combined with the usual type prior, viz., Normal prior for the mean parameter and correct prior, viz., Inverted gamma prior for the location parameter. Examples of the recently developed methodology in numerical studies are shown by the mean square error of the Bayes estimates of both parameters computed with different known and unknown nature of the parameters and results given with full discussion.

Keywords: Bayesian analysis, normal sequence; priors: Normal, Double Exponential, and Inverted gamma priors: Posterior distributions; Bayes estimates.

INTRODUCTION

A posterior analysis of the parameters of the normal sequence is discussed in this study. We can describe posterior analysis as having to first collect all the available information about the parameters, study, quantify, and express the same in the form of a probability distribution, known as a prior distribution. All information about sample observations is called the likelihood function. To get the posterior, likelihood is incorporated with the prior information. As a result, we are able to determine the unknown parameters' posterior distribution. In the case of normal sequence, many authors, including Jeffery (1961), Laplace (1812), etc., use normal prior for the mean parameter and uniform prior for the variance parameter. The General Linear Econometric Model (GLEM) problem was studied by Bhattacharya and Rain Lal (1971) with a half-normal prior for variance parameter, and the normal sequence problem was studied by Mitchell (1994) with a double exponential prior for mean parameter when variance is known. But in both cases, closed-form integral solutions were not obtained. Numerical integration technique is the only possibility, and too complicated integral equations are involved. In that case, it has a lot of complications to get an integral solution for obtaining Bayes estimates. But we assumed that one novel type of prior, such as double exponential priors mixed with the usual types of prior, viz., normal prior for the mean parameter and proper prior, viz., inverted gamma prior for location parameter, has been considered for our study for obtaining Bayes estimates.

MIXTURE OF PRIORS - NORMAL WITH DOUBLE EXPONENTIAL ALONG WITH PROPER PRIOR VIZ INVERTED GAMMA PRIOR

In this section Bayes Estimates are obtained for location parameter which follows mixture of priors distribution of Normal with Double Exponential and scale parameter which follows proper prior i.e. Inverted gamma prior.

Situation (i): Unknown variance σ^2 and unknown mean μ

Assume that X = (x_1, x_2, ..., x_n) is a random sample taken from a normal population with known variance (σ^2) and an unknown mean (μ).

That is, for i= 1,2, ...,n $f(x_i) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{1}{2\sigma^2})(x_i-\mu)^2), -\infty < \mu < \infty, 0 < \sigma < \infty$,

The following is the likelihood function of

$$P(x/\mu) \propto \frac{1}{\sigma^n}\exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2+n(\mu-\bar{x})^2\} \quad \text{... (1)}$$

The preceding distribution of μ is

$$P(\mu) = \frac{p\sqrt{\tau}}{\sqrt{2\pi}\sigma}\exp\left[\frac{-\tau}{2\sigma^2}(\mu-\theta)^2\right] + \frac{1-p}{v\sqrt{2}}\exp\{-\frac{\sqrt{2}}{v}|\mu-\theta|\} \qquad \alpha \quad \text{... (2)}$$

Where θ, v, τ > 0.

On multiplying equation(1) and (2), we get the Posterior distribution of μ

$$P(\mu/x) = \frac{1}{\sqrt{2\pi}\sigma^n}\exp\left\{-\frac{w}{2\sigma^2}\right\}\left(\frac{p\sqrt{\tau}}{\sqrt{2\pi}\sigma}\exp\left[\frac{-\tau}{2\sigma^2}(\mu-\theta)^2\right] + \frac{(1-p)}{v\sqrt{2}}\exp\{-\frac{\sqrt{2}}{v}|\mu-\theta|\}\right)$$

$$P(\mu/x) = \frac{f\,p(1-p)}{2\pi v\sigma^{n+1}}\sqrt{\frac{\tau}{2}}\exp\left(\frac{-ns^2-(\tau+n)(\mu-\mu')^2}{2\sigma^2}\right) + \frac{\tau n}{n+\tau}(\bar{x}-\theta)^2 \quad \text{...(3)}$$

Where $\mu' = \frac{n\bar{x}+\tau\theta}{\tau+n}$, $f = 1-\frac{\sqrt{2}}{v}(\bar{x}-\theta)-\frac{1}{v^2}(\bar{x}-\theta)^2$

Estimates from Bayes

After normalizing the distribution indicated in equation (3), the posterior distribution of μ is found and is given by,

$$P(\mu/x) = \frac{P(\mu/x)}{\int_{-\infty}^{\infty}P(\mu/x)d\mu}$$

$$P(\mu/x) = \frac{1}{s}\left[\exp(-\frac{1}{2\sigma^2}(\tau+n)(\mu-\mu')^2\right]\left(1-\frac{\sqrt{2}}{v}|\mu-\theta|-\frac{1}{v^2}(\mu-\theta)^2\right.$$

Where $s = \frac{\sqrt{2\pi}\,\sigma}{\sqrt{\tau+n}}\left[1-\frac{\sqrt{2}}{v}(\mu'-\theta)-\frac{1}{v^2}\left(\frac{\sigma^2}{n}+(\mu'-\theta)^2\right)\right]$

The posterior mean of μ may be found using

$$E(\mu) = \int_{-\infty}^{\infty} \mu\, P(\mu/x)\, d\mu,$$

$$= \frac{1}{s}\frac{\sqrt{2\pi}\,\sigma}{\sqrt{\tau+n}}\left[(\mu')^2 - \frac{\sqrt{2}}{v}2\theta\left(\mu'-\theta\right)^2 + \frac{1}{v^2}\left(3\mu'\frac{\sigma^2}{n+\tau} + (\mu')^3\right) + \left(\mu'-\theta\right)^2 - 2\theta\left(\frac{\sigma^2}{n+\tau} + (\mu')^2\right)\right] \quad \textbf{...(4)}$$

$$E(\mu^2) = \int_{-\infty}^{\infty} \mu^2\, P(\mu/x)\, d\mu$$

$$= \frac{1}{s}\frac{\sqrt{2\pi}\,\sigma}{\sqrt{\tau+n}}\left[\frac{\sigma^2}{n+\tau} + (\mu')^2 - \frac{\sqrt{2}}{v}\left(3\mu'\frac{\sigma^2}{n+\tau} + (\mu')^3 + \theta^2\left(\mu'-\theta\right)^2 + \frac{1}{v^2}\left(6(\mu')^2\frac{\sigma^2}{n+\tau} + (\mu')^4 + \frac{\sigma^4}{\sqrt{\tau+n}} + \frac{2\sigma^4}{\sqrt{\tau+n}} + \theta^2\left(\frac{\sigma^2}{n+\tau} + (\mu')^2\right) - 2\theta\left(3\mu'\frac{\sigma^2}{n+\tau} + (\mu')^3\right)\right]\right. \quad \textbf{...(5)}$$

Using equation (4) and (5) we get v(μ) as

$$v(\mu) = \frac{1}{s}\frac{\sqrt{2\pi}\,\sigma}{\sqrt{\tau+n}}\left(\frac{\sigma^2}{n+\tau} + (\mu')^2 - \frac{\sqrt{2}}{v}\left(3\mu'\frac{\sigma^2}{n+\tau} + (\mu')^3 + \theta^2\left(\mu'-\theta\right)^2 + \frac{1}{v^2}\left(6(\mu')^2\frac{\sigma^2}{n+\tau} + (\mu')^4 + \frac{\sigma^4}{(n+\tau)^2} + \frac{2\sigma^4}{(n+\tau)} + \theta^2\left(\frac{\sigma^2}{n+\tau} + (\mu')^2 - 2\theta\left(3\mu'\frac{\sigma^2}{n+\tau} + (\mu')^3\right) - \left(\frac{1}{s}\frac{\sqrt{2\pi}\,\sigma}{\sqrt{\tau+n}}\left\{(\mu')^2 - \frac{\sqrt{2}}{v}2\theta\left(\mu'-\theta\right)^2 + \frac{1}{v^2}\left(3\mu'\frac{\sigma^2}{n+\tau} + (\mu')^3\right) + \theta^2\left(\mu'-\theta\right)^2 - 2\theta\left(\frac{\sigma^2}{n+\tau} + (\mu')^2\right\}\right)^2\right.\right.\right.\right.$$

Where s is mentioned in equation (3)

Situation (ii): known variance σ^2 and known mean μ

Assume that $X = (x_1, x_2,...,x_n)$ is a random sample taken from a normal population with variance σ2 (unknown) and mean μ (known).i.e.,

$$f(x_i) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2\sigma^2}(x_i-\mu)^2\right), \ -\infty < \mu < \infty,\ 0 < \sigma < \infty, \text{ for } i = 1, 2,\ldots,n.$$

Given P (x/μ), the likelihood function is given by

$$P(x/\sigma) = \frac{1}{\sqrt{2\pi}\sigma^n}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\mu-\bar{x})^2\right\} \quad \textbf{...(6)}$$

Assuming an inverse gamma distribution, the previous distribution of σ can be found as

$$P(\sigma) = 2\,\frac{\beta^\alpha}{\lceil\alpha}\,(\sigma)^{-(2\alpha+1)}\exp\left(\frac{-\beta}{\sigma^2}\right), \quad \textbf{...(7)}$$

Where α, β > o, o < σ < ∞.

By multiplying equation (6) and (7) we get the Posterior distribution of σ and is given by,

$$P(\sigma/x) = \frac{1}{\sqrt{2\pi}\sigma^n}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\mu-\bar{x})^2\right\} 2\,\frac{\beta^\alpha}{\lceil\alpha}\,(\sigma)^{-(2\alpha+1)}\exp\left(\frac{-\beta}{\sigma^2}\right)$$

$$P(\sigma/x) = \sqrt{\frac{2}{\pi}}\,\frac{\beta^\alpha}{\lceil\alpha}\,(\sigma)^{-(2\alpha+n+1)}\exp\left(\frac{-\beta'}{\sigma^2}\right) \quad \textbf{...(8)}$$

Where $\beta' = \frac{w+2c}{2}$, $w = \sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\mu-\bar{x})^2$

Estimates from Bayes

The posterior mean of σ can be found using

$$E(\sigma) = \int_0^{\infty} \sigma\, P(\sigma/x)\, d\sigma$$

$$= \int_0^{\infty} \sigma\sqrt{\frac{2}{\pi}}\,\frac{\beta^\alpha}{\lceil\alpha}\,(\sigma)^{-(2\alpha+n+1)}\exp\left(\frac{-\beta'}{\sigma^2}\right) d\sigma$$

$$E(\sigma) = \sqrt{\frac{1}{2\pi}}\,\frac{\beta^\alpha}{\lceil\alpha}\left[\frac{\lceil\frac{2\alpha+n-1}{2}}{(\beta')^{\frac{2\alpha+n-1}{2}}}\right], \quad \textbf{...(9)}$$

Where w and β' mentioned in equation (8).

$$E(\sigma^2) = \int_0^\infty \sigma^2 \, P(\sigma/x) \, d\sigma$$

$$= \sqrt{\frac{2}{\pi}} \frac{\beta^\alpha}{\lceil\alpha} \int_0^\infty \sigma^2 (\sigma)^{-(2\alpha+n+1)} \exp\left(\frac{-\beta'}{\sigma^2}\right) d\sigma$$

$$E(\sigma^2) = \sqrt{\frac{1}{2\pi}} \frac{\beta^\alpha}{\lceil\alpha} \left[\frac{\lceil\frac{2\alpha+n-2}{2}}{(\beta')^{\frac{2\alpha+n-2}{2}}}\right], \quad \text{... (10)}$$

$$V(\sigma) = \sqrt{\frac{1}{2\pi}} \frac{\beta^\alpha}{\lceil\alpha} \left[\frac{\lceil\frac{2\alpha+n-2}{2}}{(\beta')^{\frac{2\alpha+n-2}{2}}}\right] - \left[\sqrt{\frac{1}{2\pi}} \frac{\beta^\alpha}{\lceil\alpha} \frac{\lceil\frac{2\alpha+n-1}{2}}{(\beta')^{\frac{2\alpha+n-1}{2}}}\right]^2 \quad \text{...(11)}$$

Where β' and w are mentioned in equation (8)

Case (iii): Unknown variance σ^2 and unknown mean μ

Assume that X = (x_1, x_2,..., x_n) is a random sample taken from a normal population with an unknown variance (σ^2) and mean (μ).

i.e., $f(x_i) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{1}{2\sigma^2}(x_i - \mu)^2)$, $-\infty < \mu < \infty$, $0 < \sigma < \infty$, for i =1, 2,…,n.

The likelihood function can be found using

$$P(x/\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^n}} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \bar{x})^2 + n(\mu - \bar{x})^2\right\} \quad \text{...(12)}$$

Given that equations (2) and (6) determine the prior distributions of μ and σ, respectively, the joint prior distribution of μ and σ can be calculated as

$$P(\mu,\sigma/x) = \frac{1}{\sqrt{2\pi\sigma^n}} \exp\left\{-\frac{w}{2\sigma^2}\right\} \left(\frac{p\sqrt{\tau}}{\sqrt{2\pi\sigma}} \exp\left[\frac{-\tau}{2\sigma^2}(\mu-\theta)^2\right]\right.$$

$$\left. + \frac{1-p}{v\sqrt{2}} \exp\{-\frac{\sqrt{2}}{v}|\mu - \theta|\}\right) 2\frac{\beta^\alpha}{\lceil\alpha} (\sigma)^{-(2\alpha+1)} \exp\left(\frac{-\beta}{\sigma^2}\right)$$

$$P(\mu,\sigma/x) = \frac{1}{\sqrt{2\pi\sigma^n}} \exp\left\{-\frac{w}{2\sigma^2}\right\} \left(\frac{p\sqrt{\tau}}{\sqrt{2\pi\sigma}} \exp\left[\frac{-\tau}{2\sigma^2}(\mu-\theta)^2\right] 2\frac{\beta^\alpha}{\lceil\alpha} (\sigma)^{-(2\alpha+1)} \exp\left(\frac{-\beta}{\sigma^2}\right) + \right.$$

$$\frac{1}{\sqrt{2\pi\sigma^n}} \exp\left\{-\frac{w}{2\sigma^2}\right\} \frac{1-p}{v\sqrt{2}} \exp\{-\frac{\sqrt{2}}{v}|\mu - \theta|\}) \, 2\frac{\beta^\alpha}{\lceil\alpha} (\sigma)^{-(2\alpha+1)} \exp\left(\frac{-\beta}{\sigma^2}\right) \quad \text{...(13)}$$

One can derive the posterior distribution of μ by integrating equation (13) in relation to σ.

$$P(\mu) = p \frac{\sqrt{\tau}}{2\pi} \frac{\beta^\alpha}{\lceil\alpha} \left\{\frac{\lceil\frac{2\alpha+n+1}{2}}{(c)^{\frac{2x+n+1}{2}}}\right\} + 1 - p \frac{1}{v\sqrt{\pi}} \frac{\beta^\alpha}{\lceil\alpha} \exp\{-\frac{\sqrt{2}}{v}|\mu - \theta|\} \left\{\frac{\lceil\frac{2\alpha+n}{2}}{(d)^{\frac{2x+n}{2}}}\right\} \quad \text{...(14)}$$

Where $c = \frac{1}{2}[ns^2 + n(\mu - \bar{x})^2 + \tau(\mu - \theta)^2 + 2\beta]$

$d = \frac{1}{2}[ns^2 + n(\mu - \bar{x})^2 + 2\beta]$

One can derive the posterior distribution of σ by integrating equation (13) in relation to μ.

$$P(\sigma) \propto p\left(\frac{\sqrt{2\pi\tau}}{\pi} \frac{\beta^\alpha}{\lceil\alpha} \frac{1}{\sqrt{\tau+n}}\right) (\sigma)^{-(2\alpha+n+1)} \exp\left(\frac{-\beta'}{\sigma^2}\right) + \frac{(1-p)f}{v\pi} \frac{\beta^\alpha}{\lceil\alpha} \exp\left(-\frac{B}{\sigma^2}\right) \frac{\sqrt{2\pi}}{\sqrt{n}} + \frac{1-p}{v^3\pi n} \frac{\beta^\alpha}{\lceil\alpha} \frac{\sqrt{2\pi}}{\sqrt{n}}$$

$$\sigma^{-(2\alpha+n-2)} \exp\left(-\frac{B}{\sigma^2}\right) \quad \text{... (15)}$$

Where $\beta' = \frac{1}{2}\left[ns^2 + \frac{\tau n}{n+\tau} (\bar{x} - \theta)^2\right]$, $B = \frac{1}{2}[ns^2 + 2\beta]$ and f is mentioned in equation (3)

Estimates from Bayes

After normalizing the distribution shown in equation (4.9), the posterior distribution of σ is found and is given by

$$P(\sigma) = \frac{P(\sigma)}{\int_0^\infty P(\sigma)d\sigma}$$

$$P(\sigma) = \frac{1}{t}\Big[p\sqrt{\frac{\tau}{n+\tau}}\,\sigma^{-(2\alpha+n+1)}\exp\left(\frac{-\beta'}{\sigma^2}\right) + (1-p)\frac{f}{v\sqrt{n}}\sigma^{-(2\alpha+n)}\exp\left(\frac{-B}{\sigma^2}\right) + (1-p)\frac{1}{nv^3\sqrt{n}}\sigma^{-(2\alpha+n-2)}\exp\left(\frac{-B}{\sigma^2}\right)\Big] \quad \textbf{... (16)}$$

Where $t = \left[p\sqrt{\frac{\tau}{n+\tau}}\frac{1}{2}\frac{\Gamma\frac{2\alpha+n}{2}}{(\beta')^{\frac{2\alpha+n}{2}}} + (1-p)\frac{f}{v\sqrt{n}}\frac{\Gamma\frac{2\alpha+n-1}{2}}{(B)^{\frac{2\alpha+n-1}{2}}} + \frac{(1-p)}{nv^2}\frac{1}{2}\frac{\Gamma\frac{2\alpha+n-3}{2}}{(B)^{\frac{2\alpha+n-3}{2}}}\right]$

In equation (3), f is mentioned.

The posterior mean of σ can be found using

$$E(\sigma) = \int_0^\infty \sigma\, P(\sigma)\, d\sigma$$

$$= \frac{1}{t}\Big[p\sqrt{\frac{\tau}{n+\tau}}\frac{1}{2}\frac{\Gamma\frac{2\alpha+n-1}{2}}{(\beta')^{\frac{2\alpha+n-1}{2}}} + (1-p)\frac{f}{v\sqrt{n}}\frac{1}{2}\frac{\Gamma\frac{2\alpha+n-2}{2}}{(B)^{\frac{2\alpha+n-2}{2}}} + \frac{(1-p)}{nv^2}\frac{1}{2}\frac{\Gamma\frac{2\alpha+n-4}{2}}{(B)^{\frac{2\alpha+n-4}{2}}}\Big] \quad \textbf{... (17)}$$

$$E(\sigma^2) = \frac{1}{t}\Big[p\sqrt{\frac{\tau}{n+\tau}}\frac{1}{2}\frac{\Gamma\frac{2\alpha+n-2}{2}}{(\beta')^{\frac{2\alpha+n-2}{2}}} + (1-p)\frac{f}{v\sqrt{n}}\frac{1}{2}\frac{\Gamma\frac{2\alpha+n-3}{2}}{(B)^{\frac{2\alpha+n-3}{2}}} + \frac{(1-p)}{nv^2}\frac{1}{2}\frac{\Gamma\frac{2\alpha+n-5}{2}}{(B)^{\frac{2\alpha+n-5}{2}}}\Big] \quad \textbf{... (18)}$$

Where β' and B are mentioned in equation (15), f is mentioned in equation (3)

As usual formula we can obtained v(σ) by using above(17) and (18) equations.

CONCLUSION:

Consequently, employing a mixture of priors, the Bayes estimates for the position and scale parameters are derived. In real life, certain complex situations cannot be described with a single prior, however good it may be, and therefore a mixture of distributions as the prior was considered and Bayes estimates carried out for known and unknown cases of parameter values.

REFERENCES

[1]. Art.Kutubi,H.s. and Noo Akma Ibrahim (2009). A Bayesian estimator for the exponential distribution with Jeffery's prior extension.

[2]. Berger, James.o (1985). Statistical Decision Theory, Springer – Varlag.

[3]. Box ,G.E.P and G.C. Tiao (1973). Bayesian inference in Statistical analysis, Addison- Wesley.

[4]. Bhattacharya,S.K and Rain Lal (1985). Bayesian analysis of GLEM with Half-Normal prior, Journal of the Korean statistical Society, Vol. 14, No.2.

[5]. De Groot M.H (1970). Optimal Statistical Decision, Mc Graw Hill.

[6]. Mitchell, A.F.S (1994). A Note on Posterior moments for a normal mean with Double Exponential Prior,Journal of Royal Statistical Society(JRSS),56(4),pp.605-610.

[7]. Reuven Y.Rubinstein (1981). Simulation and the monte carlo method, John wiley and sons, Newyork.

[8]. Venkatesan.G (1994). Bayesian Analysis of Structural Change problems in linear models. The Doctoral thesis, Department of Statistics, Annamalai University, Annamalainagar, Chidhambharam. Tamil Nadu, India.

[9]. Venkatesan.G and E. Nathiya (2008). Bayesian Analysis of Normal sequence using new type of priors, Proceedings of UGC Sponsored National Seminar, "Applied Bayesian Statistical Analysis" Department of Statistics ,Govt.Arts College (Autonomous), Salem-7, Tamil Nadu, India.

[10]. Venkatesan.G and E. Nathiya (2008). Prior distribution review in Bayesian analysis, Proceedings of UGC Sponsored National Seminar, "Applied Bayesian Statistical Analysis" Department of Statistics ,Govt.Arts College (Autonomous), Salem-7, Tamil Nadu, India.

[11]. Venkatesan.G and E. Nathiya (2010). Bayesian Analysis of Normal Sequence Using Double Exponential and Semi-Normal Prior, UGC Sponsored National Seminar on "Recent Advances in Statistics and Computer Applications", Sekcio de Statistiko, Bharathiar University, Coimbatore.

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