International Journal of Scientific Research in Modern Science and Technology, ISSN: 2583-7605 (Online)
Vol. 1, Issue 1 [2022] www.ijsrmst.com

On Quotient Hypergroups and Lie Hypergroup

Prakash Chandra Srivastava
Department of Mathematics, S. D. J. P.G. College, Chandeshwar, Azamgarh, India

Abstract:

The paper has been devoted to study on Quotient Hypergroups and Lie Hypergroup. In this paper, we have studies some properties of quotient hypergroups and use action of Lie groups on a hypergroups.

Keywords: Hypergroups, Lie Group, Quotient Hypergroup.

1. Introduction

The field of hypergroups & their applications are introduced in the following papers [1-4]. The completely simple and regular semi-hypergroups are introduced in [5]. In paper [6] introduced the concept of topological hypergroups.

A hyper-operation on P be a map from P $\times$ P *to* p^*(P) [7], where p^*(P) is set of all subsets of P. (P, o) be a H*ypergroupoid.* Let A & B are subsets of P, then

(i). $A \, o \, B = \cup\{a \, o \, b \; s.t. \; a \in A, b \in B\}$,

(ii). $x \, o \, A = \{x\} o \, A$, $x \in$ P

(iii). $A \, o \, x = A \, o \, \{x\}$.

The hypergroupoid (P, o) with following axioms is said to be *hypergroup*.

(i). $a \, o \, (b \, o \, c) = (a \, o \, b) \, o \, c$, $\forall \, a, b, c \, \epsilon$ P ,

(ii). $a \, o$ P $=$ P $o \, a =$ P , $\forall \, a \, \epsilon$ P .

If $\exists \, e \, \epsilon$ P s.t. $a \, \epsilon \, a \, o \, e \cap e \, o \, a$, $\forall \, a \, \epsilon$ P, then e is called identity element.

An element a^{-1} is *inverse* of a if $e \, \epsilon \, a \, o \, a^{-1} \cap a^{-1} \, o \, a$.

Remark (1.). If each elements of P have inverse, therefore

$$(p_1 \, o \, p_2)^{-1} = {p_2}^{-1} o \, {p_1}^{-1} \; \forall \; p_1, p_2 \, \epsilon \, \text{P}.$$

Definition 1.1.: A subset S be a hypergroup of (P, o) be a sub-hypergroup if it is satisfies by following: [1]

(i) $a \, o \, b \subseteq S \;\; \forall \, a, b \in S.$

(ii) $a \, o \, S = S \, o \, a = S \; \forall \, a \in S.$

Definition 1.2.:

A sub hypergroup of S is said to be *normal,* if

(1.1) $x \, o \, H = H \, o \, x$, for all $x \in P$

and is *supernormal* if

(1.2) $x \, o \, H \, o \, x^{-1} \subseteq H$, for all $\in$ P .

Definition 1.4.:

A map f from a hypergroups (P_1, o) to $(P_2, *)$ is said to be homomorphism if

(1.3) $\forall\ x, y \in P_1,\ f\,(x\ o\ y) \subseteq f\,(x) * f\,(y)$.

Also we can defined a good homomorphism if

(1.4) $\forall\ x, y \in P_1, f\,(x\ o\ y) = f\,(x) * f\,(y)$.

2. Quotient Hypergroups:

If H is a sub hypergroup of a hypergroup of P, then the right coset space are defined by

(2.1) $P/H = \{[x] : x \in P\}$ where $[x] = x\ o\ H$.

Let us consider,

$$\otimes : P/H \times P/H \rightarrow p * (P/H)$$

where (2.2) $[x] \otimes [y] = \{[z] : z \in x\ o\ y\}\ \forall\ [x], [y] \in P/H$.

Now, we can write and prove the following theorems:

Theorem (1): If $(\mathrm{P}, \boldsymbol{o})$ is a regular-hypergroup and H is a normal-sub-hypergroup of P, then the quotient hypergroup $(\boldsymbol{P/H}, \otimes)$ be a regular hypergroup.

Proof. Using (2.2), we can write

$$(x\ o\ H) \otimes (y\ o\ H) \subseteq \frac{P}{H} \text{ and } e_{P/H} = H,$$

$\because$ H be a normal sub-hypergroup, therefore using (1.1), we can write

$$x\ o\ H \in (x\ o\ H) \otimes H \cap H \otimes (x\ o\ H).$$

Also $(x\ o\ H)^{-1} = x^{-1}\ o\ H$, so we can write

$$\begin{aligned}(x\ o\ H \otimes y\ o\ H)^{-1} &= (x\ o\ y\ oH)^{-1}\\ &= (x\ o\ y)^{-1}\ o\ H\\ &= y^{-1} o\ x^{-1}\ o\ H\\ &= (y\ o\ H)^{-1} \otimes (x\ o\ H)^{-1}.\end{aligned}$$

Hence, the quotient hypergroup $(P/H, \otimes)$ be a regular hypergroup.

Theorem (2): If $\boldsymbol{H}$ is a normal-sub-hypergroup of a hypergroup of $(\boldsymbol{P}, \boldsymbol{o})$, then the function defined by $\boldsymbol{f} : \boldsymbol{P} \rightarrow \boldsymbol{P/H}$ s.t. $\boldsymbol{f}\,(\boldsymbol{x}) = \boldsymbol{x\ o\ H}, \forall \boldsymbol{x} \in \boldsymbol{P}$, be a homomorphism.

Proof. Let H be a normal sub hypergroup, therefore using (1.1) we have

$$H\ oH = H.$$

Since the function defined by $f : P \rightarrow P/H$, such that.

$$f\,(x) = x\ o\ H, \forall x \in P$$

Let $x, y \in P$ be arbitrary. Then, we have

$$\begin{aligned}f(x\ o\ y) &= x\ o\ y\ o\ H = x\ o\ H\ o\ y\\ &= x\ o\ H\ o\ H\ o\ y\\ &= x\ oH\ o\ y\ o\ H\\ &= f\,(x) \otimes f\,(y).\end{aligned}$$

Hence the complete proof.

Theorem (3): Let H, K be normal sub-hypergroups of hypergroup $(\mathbf{P}, \mathbf{o})$ and $H \subseteq K$. Then the quotient hypergroup K/H is a sub-hypergroup of the quotient hypergroup $(P/H, \otimes)$.

Proof. Let $a, b \in K$ be arbitrary. Using (1.2) and (2.2), we have

$$(a\ o\ H) \otimes (b\ o\ H) = \{z\ o\ H : z \in a\ o\ b\} \subseteq K/H$$

and $$(a\ o\ H) \otimes K/H = \cup_{k\in K} (a\ o\ H) \otimes (k\ o\ H)$$

$$= \cup_{k\in K} \{z\ o\ H : z \in a\ o\ k\} \subseteq K/H$$

Also, we can prove $K/H \otimes (a\ oH) \subseteq K/H$.

Thus, K is a normal sub hypergroupof P, therefore

$$a\ o\ K = K\ o\ a.$$

Hence, $(a\ o\ H) \otimes K/H = K \otimes (a\ o\ H) = K/H\ \forall a \in K$.

Hence the complete proof.

3. Lie Groups on a Hypergroups:

Here, we have investigated some axioms of the action of a Lie groups on a hypergroups. Let (P, o) is hypergroup and G is a Lie groups & we defined $P \times G \to P$.

Theorem (4): If H is a Lie sub-group of G and

$$\boldsymbol{\varphi(p_1, g_1) o\ \varphi(p_2, g_2) \subseteq \varphi(p_1 o\ p_2, g_1\ g_2),\ \forall\ p_1, p_2 \in P\ \&\ g_1, g_2 \in G},$$

then range of restrictionof action on H is sub hypergroup.

Proof: Let $\varphi(P \times H) = P_o$, then we show that (P_o, o) be a closed w.r. to action of hypergroups. Let $p'_1, p'_2 \in P_o$ then

$p_1, p_2 \in P$ and $h_1, h_2 \in H$ s.t. $p'_1 = \varphi(p_1, h_1)$ and $p'_2 = \varphi(p_2, h_2)$.

Thus,

$$p'_1\ o\ p'_2 = \varphi(p_1, h_1)\ o\ \varphi(p_2, h_2) \subseteq \varphi((p_1\ o\ p_2), h_1 h_2) \subseteq P_o.$$

Next, we have to see that P_o is a sub hypergroups.

Let $\varphi(p, h) \in P_o$ therefore,

$$\varphi(p, h) \epsilon\ \varphi(p_1, \quad h_1)\ o\ \varphi(p_2, h_2) \epsilon\ \varphi(p_1\ o\ p_2, h_1 h_2) \subseteq x\ o\ P,$$

where $x = \varphi(p_1, h_1)$.

4. Conclusion

In this paper we have studies on Quotient Hypergroups and Lie Hypergroup. We have obtained some results in the form of theorems (1), (2), (3) and (4).

References:

[1] P. Corsini, V. Leoreanu, Application of hyperstructure theory, Kluwer Academic Publisher, 2003.

[2] M. Scafati Tallini, (2012) Hypergroups and geometric spaces, Ratio Mathematica, **22**, 69-84.

[3] M. Toomanian, M. Amini and A. Heydari, (2016) Representations of double coset Lie hyper groups, Iran. J. Math. Sci. Inform. 11**,** 87-96.

[4] H. Yazarli, D. Yi lmaz and B. Davvaz, (2020) The maximal hyperrings of quotients, Ital. J. Pure Appl. Math. 44, 938-951.

[5] H. Mousa Jafarabadi, N. Haniza Sarmin and M. R. Molaei, Completely simple and regular semi hypergroups, Bull. Malays. Math. Sci. Soc. 35, 335-343, (2012).

[6] D. Heidari, B. Davvaz and S. M. S Modarres, (2014) Topological hypergroups in the sense of Marty, Commun. Algebra. 42, 4712-4721.

[7] D. Heidari, B. Davvaz, S. M. S. Modarres, (2016) Topological Polygroups, Bull. Malays. Math. Sci. Soc. 39, 707-721.