NORMAL PROJECTIVE INFINITESIMAL TRANSFORMATION AND CURVATURE COLLINEATION IN A FINSLER SPACE

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ABSTRACT:

In this paper we have obtained the result as Lie derivatives of normal projective curvature tensor N^i_{jkh} and of the Ricci tensor N_{kh} and in this continuation we have derived certain more results the projective deviation tensor and its Lie derivatives.

Keywords: Normal projective infinitesimal transformation, Normal projective curvature collineation, Ricci normal projective curvature collineation, Lie-derivative, Cartan's curvature collineation, Cartan's Ricci collineation, Affine motion.

1. INTRODUCTION:

Takano [2] who discussed Riemannian space with bi-recurrent curvature of the projective motion and Yano and Nagano [7] presented projective conformal transformation in a Riemannian space and studies of curvature collineation and its properties in a Riemannian space have been carried out by Katzin, Levine and Davis [1]. Sinha [3] who defined the infinitesimal projective transformation and by Pande & Kumar [4] who also defined infinitesimal transformation $\bar{x}^i = x^i + v^i\,(x)dt$ and derived some results in the form of theorems. An attempt to extend the theory of curvature collineation in Finsler space has been made by Singh and Prasad [8] and Pande and Kumar [5]. The relations which satisfied in a Finsler space accepted by curvature collineation and others symmetry have been studied by these researchers.

2. NORMAL PROJECTIVE INFINITESIMAL TRANSFORMATION AND CURVATURE COLLINEATION

Yano [9] has defined the Lie derivatives of any tensor field $T^i_j(x,\dot{x})$ and the connection coefficient $\Pi^i_{jk}(x,\dot{x})$ with respect to the normal projective covariant derivative respectively given as:

$$\pounds_v T^i_j \;=\; (\nabla_s T^i_j)v^s - T^s_j(\nabla_s v^i) + T^i_s(\nabla_j v^s) + (\dot{\partial}_s T^i_j)(\nabla_m v^s)\dot{x}^m \tag{2.1}$$

and
$$\pounds_v \Pi^i_{jk} = \nabla_j \nabla_k\, v^i + (\dot{\partial}_h \Pi^i_{jk})(\nabla_s v^h)\dot{x}^s\,. \tag{2.2}$$

Also we using the following commutation formulae in the form of Lie-derivative and normal projective covariant derivative

$$\dot{\partial}_r\,(\pounds_v T^i_{jh}) - \pounds_v(\dot{\partial}_r T^i_{jk}) = 0\,, \tag{2.3}$$

$$\pounds_v(\nabla_r T^i_{jk}) - \nabla_r(\pounds_v T^i_{jk}) = T^s_{jk}\,\pounds_v \Pi^i_{sr} - T^i_{sk}\,\pounds_v \Pi^s_{jr} - T^i_{js}\,\pounds_v \Pi^s_{kr} - (\dot{\partial}_s T^i_{jk})(\pounds_v \Pi^s_{tr})\dot{x}^t \tag{2.4}$$

and

$$\nabla_j(\pounds_v\,\Pi^i_{kh}) - \nabla_k(\pounds_v\,\Pi^i_{jh}) = (\pounds_v N^i_{jkh}) + (\dot{\partial}_r \Pi^i_{kh})(\pounds_v \Pi^r_{tj})\dot{x}^t - (\dot{\partial}_r \Pi^i_{jh})(\pounds_v \Pi^r_{tk})\dot{x}^t \tag{2.5}$$

where, N^i_{jkh} is the normal projective curvature tensor and this curvature tensor satisfies the identities and contractions have been given as

$$v^p N^i_{kjp} = (\rho_k \delta^i_j - \rho_j \delta^i_k) + \rho(\delta^i_k \emptyset_j - \delta^i_j \emptyset_k) + (\partial_k \emptyset_j - \partial_j \emptyset_k) v^i \ . \qquad (2.6)$$

Now we give the following definitions:

DEFINITION (1):

A Finsler space F_n is defines to be an affine motion if

$$\pounds_v \Pi^i_{jk} = 0 \qquad (2.7)$$

DEFINITION (2):

A Finsler space is defines to be an affinely connected if

$$\dot{\partial}_r \Pi^i_{jk} = 0 \ . \qquad (2.8)$$

DEFINITION (3):

In a Finsler space F_n, the infinitesimal point transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines to a normal projective curvature collineation if

$$\pounds_v N^i_{jkh} = 0. \qquad (2.9)$$

DEFINITION (4):

A Finsler space F_n is defines to a Ricci normal projective curvature collineation if

$$\pounds_v N_{kh} = 0 \ . \qquad (2.10)$$

DEFINITION (5):

The infinitesimal point transformation $\bar{x}^i = x^i + v^i(x)dt$ is said to be an infinitesimal normal projective transformation in an F_n if

$$\pounds_v \Pi^i_{jk} = \delta^i_j b_k + \delta^i_k b_j + g_{jk} g^{ir} d_r \qquad (2.11)$$

where $b_j(x, \dot{x})$ and $d_r(x, \dot{x})$ are vector fields satisfying the following

$$\text{(a) } \dot{\partial}_j b = b_j \ , \quad \text{(b) } \dot{\partial}_k b_j = b_{jk} \ , \quad \text{(c) } b_{jk} \dot{x}^k = b_j \ , \qquad (2.12)$$

$$\text{(d) } b_j \dot{x}^j = b \ , \quad \text{(e) } \dot{\partial}_j d = d_j \ , \quad \text{(f) } \dot{\partial}_k d_j = d_{jk} \ ,$$

$$\text{(g) } d_{jk} \dot{x}^k = d_j \quad \text{and} \quad \text{(h) } d_j \dot{x}^j = d \ .$$

Using (2.9) in (2.5), we get

$$\pounds_v N^i_{jkh} = \delta^i_k (\nabla_j b_h) + \delta^i_h (\nabla_j b_k) + g_{kh} g^{ir} (\nabla_j d_r) - \delta^i_j (\nabla_k b_h) -$$

$$- \delta^i_k (\nabla_h b_j) - g_{jh} g^{ir} (\nabla_k d_r) - b(\dot{\partial}_j \Pi^i_{kh}) - (\dot{\partial}_m \Pi^i_{kh}) g_{rj} g^{rp} d_p \dot{x}^m . \qquad (2.13)$$

where, Kroncker delta and the two fundamental tensors have vanishing normal projective covariant derivatives.

Allowing a transvection of (2.13) by $\dot{x}^j \dot{x}^k$, we have

$$\pounds_v N^i_{jkh} \dot{x}^j \dot{x}^k = \delta^i_k (\nabla_j b_h) \dot{x}^j \dot{x}^k + \delta^i_h (\nabla_j b) \dot{x}^j + g_{kh} g^{ir} (\nabla_j d_r) \dot{x}^j \dot{x}^k -$$

$$- \delta^i_h (\nabla_k b) \dot{x}^k - g_{jh} g^{ir} (\nabla_k d_r) \dot{x}^j \dot{x}^k - (\dot{\partial}_r \Pi^i_{kh}) g_{sj} g^{rp} d_p \dot{x}^s \dot{x}^j \dot{x}^k \ , \qquad (2.14)$$

where, we have taken into account (2.12).

Now allowing a contraction in equation (2.14) w.r.t. the indices i & h, we get

$$\pounds_v N^i_{jki} \dot{x}^j \dot{x}^k = (n+1)(\nabla_j b) \dot{x}^j - (n+1)(\nabla_k b) \dot{x}^k + (\nabla_j d) \dot{x}^j - (\nabla_k d) \dot{x}^k -$$

$$- (\dot{\partial}_r \Pi^i_{kh}) g_{sj} g^{rp} d_p \dot{x}^s \dot{x}^j \dot{x}^k \,. \qquad (2.15)$$

Now allowing contraction (2.13) with respect to the indices i and j and thereafter using (2.6) we get

$$\pounds_v N_{kh} = \delta^i_k (\nabla_i b_h) + \delta^i_h (\nabla_i b_k) + g_{kh} g^{ir} (\nabla_i d_r) - n(\nabla_k b_h) -$$
$$- \delta^i_k (\nabla_h b_i) - \delta^r_h (\nabla_k d_r) - b(\dot{\partial}_i \Pi^i_{kh}) - \delta^p_i (\dot{\partial}_m \Pi^i_{kh}) d_p \dot{x}^m. \quad (2.16)$$

We eliminating the term $(\nabla_k b)\dot{x}^k$ from (2.14) & (2.15), we get

$$Q^i_h = (n+1)[(\nabla_j b_h)\dot{x}^i \dot{x}^j + g_{kh} g^{ir} (\nabla_j d_r) \dot{x}^j \dot{x}^k - (\nabla_k b_h) \dot{x}^i \dot{x}^k$$
$$- g_{jh} g^{ir} (\nabla_k d_r) \dot{x}^j \dot{x}^k - (\dot{\partial}_r \Pi^i_{kh}) g_{sj} g^{rp} d_p \dot{x}^s \dot{x}^j \dot{x}^k] +$$
$$+ (\dot{\partial}_r \Pi^i_{kh}) g_{sj} g^{rp} d_p \dot{x}^s \dot{x}^j \dot{x}^k \,. \qquad (2.17)$$

where $Q^i_h = (n+1) \pounds_v N^i_{jkh} \dot{x}^j \dot{x}^k - \delta^i_h \pounds_v N^i_{jki} \dot{x}^j \dot{x}^k$.

We consider the projective deviation tensor $W^i_j(x, \dot{x})$ as has been given

$$W^i_j = H^i_j - H\, \delta^i_j - \frac{1}{n+1} (\dot{\partial}_k H^k_j - \dot{\partial}_j H)\, \dot{x}^i$$

apply the commutation formula (2.4) to this deviation tensor and get

$$\pounds_v(\nabla_r W^i_j) - \nabla_r(\pounds_v W^i_j) = W^s_j \pounds_v \Pi^i_{sr} - W^i_s \pounds_v \Pi^s_{jr} - (\dot{\partial}_s W^i_j)(\pounds_v \Pi^s_{tr}) \dot{x}^t \,. \qquad (2.18)$$

Using (2.11) in (2.18), we get

$$\pounds_v(\nabla_r W^i_j) - \nabla_r(\pounds_v W^i_j) = W^s_j (\delta^i_s b_r + \delta^i_r b_s + g_{sr} g^{ip} d_p) - W^i_s (\delta^s_j b_r + \delta^s_r b_j +$$
$$+ g_{jr} g^{sp} d_p) - (\dot{\partial}_s W^i_j)(\delta^s_t b_r + \delta^s_r b_t + g_{tr} g^{sp} d_p) \dot{x}^t \,. \qquad (2.19)$$

Now we make the assumption that $\pounds_v(\nabla_r W^i_j) = 0$, then from (2.19), we have

$$\nabla_r(\pounds_v W^i_j) = (\dot{\partial}_s W^i_j) b_r \dot{x}^s + b(\dot{\partial}_r W^i_j) + (\dot{\partial}_s W^i_j)\, g_{tr} g^{sp} d_p \dot{x}^t + W^i_r\, b_j +$$
$$+ W^i_s\, g_{jr} g^{sp} d_p - W^s_j \delta^i_r b_s - W^s_j\, g_{sr} g^{ip} d_p \,. \qquad (2.20)$$

We now allowing a contraction in equation (2.20) w.r.t. the indices i & r, we get

$$\nabla_i(\pounds_v W^i_j) = (\dot{\partial}_s W^i_j) b_i \dot{x}^s + (\dot{\partial}_s W^i_j) g_{ti} g^{sp} d_p \dot{x}^t + W^i_s g_{ji} g^{sp} d_p - n\, W^s_j b_s - W^s_j d_s \,. \quad (2.21)$$

Allowing a transvection in (2.20) by $\dot{x}^r$, we get

$$\nabla_r(\pounds_v W^i_j)\dot{x}^r = (\dot{\partial}_s W^i_j) b\, \dot{x}^s + b\, (\dot{\partial}_r W^i_j)\dot{x}^r + (\dot{\partial}_s W^i_j)\, g_{tr} g^{sp} d_p \dot{x}^t \dot{x}^r + W^i_r b_j \dot{x}^r +$$
$$+ W^i_s\, g_{jr} g^{sp} d_p \dot{x}^r - W^s_j b_s \dot{x}^r - W^s_j\, g_{sr} g^{ip} d_p \dot{x}^r \,, \qquad (2.22)$$

while writing (2.22), we have taken into account

(a) $W^i_j \dot{x}^r = 0$, (b) $\dot{\partial}_h W^i_r\, \dot{x}^r = - W^i_h$, (c) $\dot{\partial}_i W^i_j = 0$,

(d) $\dot{\partial}_j W^i_k\, \dot{x}^j = 2W^i_k$, (e) $W^i_i = 0$. (2.23)

We now eliminating the term $W^s_j b_s$ with the help of (2.21) and (2.22) and after get

$$n\, \nabla_r(\pounds_v W^i_j)\dot{x}^r - \nabla_r(\pounds_v W^i_j)\dot{x}^r = R^i_j + S^i_j \,, \qquad (2.24)$$

where $R^i_j = n\, [(\dot{\partial}_s W^i_j) b\, \dot{x}^s + b\, (\dot{\partial}_r W^i_j)\dot{x}^r + (\dot{\partial}_s W^i_j)\, g_{tr} g^{sp} d_p \dot{x}^t \dot{x}^r -$

$$- W^s_j\, g_{sr} g^{ip} d_p \dot{x}^r] \qquad (2.25)$$

and $S^i_j = [(\dot{\partial}_s W^i_j) b\, \dot{x}^s + (\dot{\partial}_s W^i_j)\, g_{tr} g^{sp} d_p \dot{x}^r \dot{x}^t - W^t_s g_{jt} g^{sp} d_p \dot{x}^i]$. (2.26)

Now in view of given definitions we shall summarize all those observations and we have obtained the result in the form of theorems. Keeping in mind the definition (2.1) and given by (2.6), from (2.11) we arrive at the conclusion that if the infinitesimal transformation $\bar{x}^i = x^i + v^i(x)dt$ defines an affine motion then the covariant vector fields $b_j(x,\dot{x})$ and $d_r(x,\dot{x})$ should vanish. Therefore, we can state:

THEOREM (1):

If the infinitesimal normal projective transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines an affine motion in a Finsler space F_n, then the covariant vector fields $b_j(x,\dot{x})$ and $d_r(x,\dot{x})$ appearing in (2.11) should vanish.

Now we consider that the infinitesimal transformation $\bar{x}^i = x^i + v^i(x)dt$ defines a normal projective curvature collineation then in this case using (2.9) and (2.13), then we can state:

THEOREM (2):

If the infinitesimal normal projective point transformation $\bar{x}^i = x^i + v^i(x)dt$ is states a normal projective curvature collineation, then we shall have

$$\delta^i_k(\nabla_j b_h) + \delta^i_h(\nabla_j b_k) + g_{kh}\, g^{ir}(\nabla_j d_r) - \delta^i_j(\nabla_k b_h) - \delta^i_k(\nabla_h b_j) -$$

$$- g_{jh}\, g^{ir}(\nabla_k d_r) - b(\dot{\partial}_j \Pi^i_{kh}) - (\dot{\partial}_m \Pi^i_{kh}) g_{rj}\, g^{rp} d_p \dot{x}^m = 0 . \quad (2.27)$$

Now if we assume that the covariant vector fields $b_j(x,\dot{x})$ and $d_r(x,\dot{x})$ appearing in the equation (2.11) be normal projective covariant constants, then in this case using (2.13) we can state:

THEOREM(3):

If the covariant vector fields $b_j(x,\dot{x})$ and $d_r(x,\dot{x})$ appearing in (2.11) are projective covariant constants in a Finsler space F_n, then we have

$$£_v N^i_{jkh} + b(\dot{\partial}_j \Pi^i_{kh}) + (\dot{\partial}_m \Pi^i_{kh}) g_{rj} g^{rp} d_p \dot{x}^m = 0 . \quad (2.28)$$

If we assume that in the case of normal projective curvature collineation then we get:

$$b(\dot{\partial}_j \Pi^i_{kh}) + (\dot{\partial}_m \Pi^i_{kh}) g_{rj} g^{rp} d_p \dot{x}^m = 0 \quad (2.29)$$

where, the covariant vector fields $b_j(x,\dot{x})$ and $d_r(x,\dot{x})$ presence in (2.11) be taken to be normal projective covariant constant.

Now we consider the case when the infinitesimal point transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines a normal projective Ricci collineation then in this case we can state with the help of (2.10) and (2.16):

THEOREM (4):

in a Finsler space F_n, If the infinitesimal normal projective transformation $\bar{x}^i = x^i + v^i(x)dt$ is states a normal projective Ricci collineation, then we have

$$(1-n)(\nabla_k b_h) + g_{kh}\, g^{ir}(\nabla_j d_r) - (\nabla_k b_h) - b(\dot{\partial}_i \Pi^i_{kh}) - (\dot{\partial}_m \Pi^i_{kh}) d_i \dot{x}^m = 0 . \quad (2.30)$$

3. CONCLUSION :

In this paper we have obtained results in the form of Lie derivatives of normal projective curvature tensor N^i_{jkh} and of the Ricci tensor N_{kh} and in this continuation we have derived certain more results the projective deviation tensor and its Lie derivatives. After these observations we have derived obtained results in the form of theorems telling as to what will happen to the covariant vector fields $b_j(x,\dot{x})$ and $d_r(x,\dot{x})$ when the infinitesimal normal projective transformation $\bar{x}^i = x^i + v^i\ (x)dt$ defines an affine motion and also derived the results which will hold when the infinitesimal normal projective point transformation defines a normal projective curvature and normal projective Ricci collineations.

REFERENCES

[1] Katzin, G.H. Levine, J and Davis, W.A. : Curvature collineations: a fundamental symmetry property of the space time of general relativity defined by the vanishing of the Lie derivative of the Riemannian curvature tensor, Jour. of Math. Phys. 10, 617-629, (1969).

[2] Takano, K. : On projective motion in spaces with bi-recurrent curvature Tensor (N.S.), 12, 28-32, (1962).

[3] Sinha, R.S. : Infinitesimal projective transformation in a Finsler space, Prog. Math. 5(1-2), 30-39, (1971).

[4] Pande, H.D. and Kumar, A. : Special infinitesimal projective transformation in a Finsler space, Acad. Naz. dei Lincei Rend., 58(3-4), 190-193, (1974).

[5] Pande, H.D and Kumar, A. : Special curvature collineation and projective symmetry in a Finsler space, Acad. Naz. dei Lincei Rend. Ser. XI, 52(1), 37-45, (1973).

[6] Rund. H. : The differential geometry of Finsler spaces, Springer Verlag, Berlin, (1957).

[7] Yano, K. and Nagano, T. : Some theorems on projective conformal transformation, Inag.Math.19, 451-459, (1959).

[8] Singh, U.P. and Prasad B.N. : Special curvature collineation in a Finsler space, Acad. Naz. dei Lincei Rend. Ser. VIII, 50(2), 82-87, (1971).

[9] Yano, K. : The theory of Lie derivatives and its applications, North Holland Publ. Co., Amsterdam (1957).