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On Extended Transformation Formulae Associated to the k-Hypergeometric Function

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ABSTRACT

Recently the second order homogeneous k-hypergeometric differential equation and the definition of the function ${}_2\mathcal{F}_1[(\mathrm{a},\mathrm{k}),(b,k);(c,k);z]$ as its solution at the origin have been presented. Motivated by the current research work in the study of k-hypergeometric functions, our aim is to establish certain extended transformation formulae related to the k-hypergeometric function with the Euler's identity for Gauss hypergeometric function as a special case.

Keywords: k-pochhammer symbol, k-hypergeometric functions, hypergeometric transformation.

Subject Classification: [2020]33C05, 34A25

1. Introduction

The Gauss hypergeometric differential equation

$$z(1-z)\frac{d^2y}{dz^2}+[c-(1+a+b)z]\frac{dy}{dz}-aby=0 \qquad (1.1)$$

and its solutions near the regular singular points 0, 1and ∞ were studied by many authors such as Coddington[3], Campos [4], Gasper [6], RainVille [11], Slater [12], Whittaker [14] .

Using Frobenius method, we can find a convergent hypergeometric series solution

$${}_2\mathcal{F}_1[a,b;c;z]=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

of equation (1.1) in

$R=\{z:|z|<1\}, when\ c\notin z, z^{1-c}{}_2\mathcal{F}_1[\mathrm{a-c+1,b-c+1;2-c;z}]$

is second solution of equation(1.1) in R. Recently the equation of the form

$$kz(1-kz)\frac{d^2y}{dz^2}+[c-(k+a+b)kz]\frac{dy}{dz}-aby=0 \qquad (1.2)$$

has been proposed and using the extended Pochhammer symbol its Frobenius solution

$$y_1(z)={}_2\mathcal{F}_1\,[(\mathrm{a},\mathrm{k}),(b,k);(c,k);z]=\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}}\frac{z^n}{n!}c\notin N_0$$

is defined as the k-hypergeometric function.

Further the second solution of equation (1.2) has been deduced as

$y_2(z) = z^{1-\frac{c}{k}} {}_2\mathcal{F}_1 \left[(\mathrm{a}+\mathrm{k}-\mathrm{c},\mathrm{k}),(b+k-c,k);(2k-c,k);z\right]$ for $c-2k \notin N_0$.

Some works and references concerning the above theoretical developments can be found in Ahmad[1], Ali[2],Krasniqi [7], Mubeen [8-10], Shengfeng [13].

In the present paper by substitution in equation (1.2) and using above solutions, two extended transformation formulae have been derived as a extended form of Euler's identity.

2. Preliminaries

This section deals with the basic definitions and facts related to the extended hypergeometric differential equation and k-hypergeometric series solutions based on the research of Diaz et al. [5], Krasniqi [7], Mubeen [8-10], Shengfeng [13], Yilmazer[15].

Definition 2.1: For the standard form of differential equation :

$$\frac{d^2y}{dz^2}+p(z)\frac{dy}{dz}+q(z)y(z)=0 \qquad (2.1)$$

If $z=z_0$ is a singularity of equation (2.1), then $z=z_0$ is a regular singularity of (2.1) iff $(z-z_0)p(z),(z-z_0)^2q(z)$ are analytic in $\{z:|z-z_0|<R\}$, R >0.

***Definition 2.2*:**

The Pochhammer k-symbol $(\sigma)_{n,k}$ is defined by:

$$(\sigma)_{n,k}=\sigma(\sigma+k)(\sigma+2k)\dots(\sigma+(n-1)k),\sigma\neq 0 \qquad (2.2)$$

$(\sigma)_{0,k}=1$ where $k>0$

***Definition 2.3*:** The k-hypergeometric series as a function of parameters is defined as:

$${}_2\mathcal{F}_1\left[(\mathrm{a},\mathrm{k}),(b,k);(c,k);z\right]=\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}}\frac{z^n}{n!} \qquad (2.3)$$

3. Main Results

Theorem 3.1: The following transformation formula for k-hypergeometric function holds:

$${}_2\mathcal{F}_1[(a,k),(b,k);(c,k);z]=\{(1-kz)\}^{-\frac{a}{k}}\ {}_2\mathcal{F}_1\left[(\mathrm{a},\mathrm{k}),(c-b,k);(c,k);\frac{-z}{1-kz}\right]$$

Proof: Let $\quad y=\{k(1-kz)\}^{-\frac{a}{k}}w \qquad (3.1)$

$$\frac{dy}{dz}=\{k(1-kz)\}^{-\frac{a}{k}}\frac{dw}{dz}+ak(k-k^2z)^{-\frac{a}{k}-1}w$$

$$\frac{d^2y}{dz^2}=\{k(1-kz)\}^{-\frac{a}{k}}\frac{d^2w}{dz^2}+2ak(k-k^2z)^{-\frac{a}{k}-1}\frac{dw}{dz}+ak(a+k)k(k-k^2z)^{-\frac{a}{k}-2}w$$

On plunging the above terms into the k-analogues hypergeometric differential equation (1.2) we get,

$$kz(1-kz)\left[\frac{d^2w}{dz^2}+2ak(k-k^2z)^{-1}\frac{dw}{dz}+a(a+k)k^2(k-k^2z)^{-2}w\right]+$$
$$+\{c-(k+a+b)kz\}\left[\frac{dw}{dz}+ak(k-k^2z)^{-1}w\right]-abw=0$$

$$kz(1-kz)^2\frac{d^2w}{dz^2}+(1-kz)[c+(a-b-k)kz]\frac{dw}{dz}+a(c-b)w=0 \qquad (3.2)$$

Now, let

$t = \frac{-z}{(1-kz)}, and\ then\ kz = \frac{-kt}{(1-kt)}, 1 - kz = \frac{1}{(1-kt)}$

$$\frac{dw}{dz} = -(1-kt)^2 \frac{dw}{dt}$$

$$\frac{d^2w}{dz^2} = \left[(1-kt)^4 \frac{d^2w}{dt^2} - 2k(1-kt)^3 \frac{dw}{dt}\right]$$

Equation (3.2) becomes

$$\left(\frac{-kt}{(1-kt)}\right)\left(\frac{1}{(1-kt)}\right)^2 \left[(1-kt)^4 \frac{d^2w}{dt^2} - 2k(1-kt)^3 \frac{dw}{dt}\right] + \left(\frac{1}{(1-kt)}\right)[c + (a-b-k)]\left(\frac{-kt}{(1-kt)}\right)\left(-(1-kt)^2 \frac{dw}{dt}\right) + a(c-b)w = 0$$

$$kt(1-kt)\frac{d^2w}{dt^2} + [c - (k+a+c-b)kt]\frac{dw}{dt} - a(c-b)w = 0 \qquad (3.3)$$

Which is a k-hypergeometric type differential equation with different coefficient and whose solution can be written as

$$w = {}_2\mathcal{F}_1[(a,k),(c-b,k);(c,k);t]$$

$$y = \{k(1-kz)\}^{-\frac{a}{k}} {}_2\mathcal{F}_1\left[(\mathrm{a},\mathrm{k}),(c-b,k);(c,k);\frac{-z}{1-kz}\right]$$

Now we have two linearly independent solution of equation (1.2) as

$$y_1(z) = {}_2\mathcal{F}_1[(a,k),(b,k)(c,k);z]$$

$$y_2(z) = z^{\left(1-\frac{c}{k}\right)} {}_2\mathcal{F}_1[(k+a-c,k),(k+b-c,k);(2k-c,k);z]$$

So, as usual we have $y = Ay_1 + By_2$

And by letting $k = 1\ and\ z \to 0$, we conclude $B = 0\ and\ A = (k)^{-\frac{a}{k}}, so\ that\ if$

$$|z| < \frac{1}{k}\ and\ |1-kz| < 1,$$

$${}_2\mathcal{F}_1[(a,k),(b,k);(c,k);z] = \{(1-kz)\}^{-\frac{a}{k}} {}_2\mathcal{F}_1\left[(\mathrm{a},\mathrm{k}),(c-b,k);(c,k);\frac{-z}{1-kz}\right]$$

Theorem 3.2. If $|z| < \frac{1}{k}\ and\ \left|\frac{z}{1-kz}\right| < \frac{1}{k}$, then the following transformation formula for k-hypergeometric function holds:

$${}_2\mathcal{F}_1\,[(a,k),(b,k);(c,k);z] = (1-\mathrm{kz})^{\frac{1}{k}(c-a-b)} {}_2\mathcal{F}_1[(\mathrm{c}-\mathrm{a},\mathrm{k}),(c-b,k);(c,k);z] \qquad (3.4)$$

Proof: Consider

$${}_2\mathcal{F}_1[(\mathrm{a},\mathrm{k}),(c-b,k);(c,k);w] = (1-\mathrm{k}w)^{-\frac{(\mathrm{c}-\mathrm{b})}{k}} {}_2\mathcal{F}_1\left[(\mathrm{c}-\mathrm{b},\mathrm{k}),(c-a,k);(c,k);-\frac{w}{(1-kw)}\right]$$

On taking

$$w = -\frac{z}{1-kz} \quad i.e. \quad z = -\frac{w}{1-kw}\ and\ (1-kw) = (1-kz)^{-1}$$

$${}_2\mathcal{F}_1[(\mathrm{a},\mathrm{k}),(c-b,k);(c,k);w] = (1-\mathrm{kz})^{\frac{(\mathrm{c}-\mathrm{b})}{k}} {}_2\mathcal{F}_1[(\mathrm{c}-\mathrm{b},\mathrm{k}),(c-a,k);(c,k);z]$$

$${}_2\mathcal{F}_1\left[(\mathrm{a},\mathrm{k}),(c-b,k);(c,k);-\frac{z}{1-kz}\right] = (1-\mathrm{kz})^{\frac{(\mathrm{c}-\mathrm{b})}{k}} {}_2\mathcal{F}_1[(\mathrm{c}-\mathrm{b},\mathrm{k}),(c-a,k);(c,k);z] \qquad (3.5)$$

Multiplying both side of (3.5) by $(1-kz)^{-\frac{a}{k}}$ we get

$$(1-kz)^{-\frac{a}{k}}\,{}_2\mathcal{F}_1\left[(a,k),(c-b,k);(c,k);-\frac{z}{1-kz}\right]=(1-kz)^{\frac{(c-a-b)}{k}}{}_2\mathcal{F}_1[(c-b,k),(c-a,k);(c,k);z]$$

and using theorem (3.1) we get

$${}_2\mathcal{F}_1[(a,k),(b,k);(c,k);z]=(1-kz)^{\frac{1}{k}(c-a-b)}{}_2\mathcal{F}_1[(c-a,k),(c-b,k);(c,k);z]$$

(Extended form of Euler' identity for k-hypergeometric functions)

Corollary 3.3.: If we take $k=1$ in (3.4), we have

$${}_2\mathcal{F}_1[a,b;c;z]=(1-z)^{(c-a-b)}{}_2\mathcal{F}_1[c-a,c-b;c;z] \qquad (3.6)$$

This transformation relation of usual hypergeometric function is also known as Euler's identity.

Conclusion:

In this paper, I have derived the extended transformation relations involving the extended hypergeometric functions, which can be useful in deduction of solutions of k-hypergeometric differential equation (1.2) like the Kummer's twenty-four solutions of the Gauss hypergeometric differential equation. Since the results investigated here are general in nature, it is expected that the results will be useful addition in the theory of the Gauss hypergeometric functions.

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